

## THE SPECTRAL SHIFT FUNCTION AND SPECTRAL FLOW.

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**Abstract.** At the 1974 International Congress, I. M. Singer proposed that eta invariants and hence spectral flow should be thought of as the integral of a one form. In the intervening years this idea has lead to many interesting developments in the study of both eta invariants and spectral flow. Using ideas of [24] Singer's proposal was brought to an advanced level in [16] where a very general formula for spectral flow as the integral of a one form was produced in the framework of noncommutative geometry. This formula can be used for computing spectral flow in a general semifinite von Neumann algebra as described and reviewed in [5]. In the present paper we take the analytic approach to spectral flow much further by giving a large family of formulae for spectral flow between a pair of unbounded self-adjoint operators  $D$  and  $D + V$  with  $D$  having compact resolvent belonging to a general semifinite von Neumann algebra  $\mathcal{N}$  and the perturbation  $V \in \mathcal{N}$ . In noncommutative geometry terms we remove summability hypotheses. This level of generality is made possible by introducing a new idea from [3]. There it was observed that M. G. Krein's spectral shift function (in certain restricted cases with  $V$  trace class) computes spectral flow. The present paper extends Krein's theory to the setting of semifinite spectral triples where  $D$  has compact resolvent belonging to  $\mathcal{N}$  and  $V$  is any bounded self-adjoint operator in  $\mathcal{N}$ . We give a definition of the spectral shift function under these hypotheses and show that it computes spectral flow. This is made possible by the understanding discovered in the present paper of the interplay between spectral shift function theory and the analytic theory of spectral flow. It is this interplay that enables us to take Singer's idea much further to create a large class of one forms whose integrals calculate spectral flow. These advances depend critically on a new approach to the calculus of functions of non-commuting operators discovered in [3] which generalizes the double operator integral formalism of [8, 9, 10]. One surprising conclusion that follows from our results is that the Krein spectral shift function

is computed, in certain circumstances, by the Atiyah-Patodi-Singer index theorem [2].

## INTRODUCTION

**Overview.** In [3] we gave an analytic proof that the spectral shift function of M. G. Krein computes the spectral flow under certain restricted circumstances. Spectral flow stems from the work of Atiyah-Patodi-Singer [2] where it is introduced primarily in a topological sense. Subsequently, starting with a suggestion of I. M. Singer at the 1974 Vancouver ICM, the idea that spectral flow could be expressed as the integral of a one form has been extensively studied in the framework of unbounded Fredholm modules (or spectral triples) beginning with [24] and continuing in [15, 16]. The current state of knowledge using an analytic approach due to Phillips [30, 31] is described in detail in [5].

On the other hand the Krein spectral shift function has an extensive history in perturbation theory which may be partly traced from [26, 6, 11, 32]. The spectral shift theory was developed by both physicists and mathematicians in the context of perturbation theory of Schrodinger operators. By the Birman-Krein formula, it is related to the phase of the scattering matrix in the latter's spectral representation. Krein's formula for the spectral shift function, which is the motivation for this paper, is restricted to the case where one perturbs an arbitrary unbounded self adjoint operator  $D_0$  by a trace class operator. The spectral shift function compares in a sense the relation between the spectrum of  $D_0$  and that of its perturbation. In this paper we will extend the theory of the spectral shift function to the situation where we perturb  $D_0$  by an arbitrary bounded self adjoint operator but adopt the assumption that  $D_0$  has compact resolvent motivated by the notion of spectral triple. One surprising conclusion that follows is that the Atiyah-Patodi-Singer (or APS) index theorem [2] and its generalizations are, in certain instances, computing the spectral shift function.

The generalizations of the APS index theorem we are referring to in the previous paragraph have to do with situations where von Neumann algebras other than all the bounded operators on a Hilbert space arise. The standard theory of spectral flow deals with unbounded self adjoint Fredholm operators. Using an approach due to Phillips *op cit* spectral flow may be defined analytically (but as yet not topologically) for certain operators affiliated to semifinite von Neumann algebras, the so-called Breuer-Fredholm theory, which is reviewed in [5]. In addition the connection between spectral flow in semifinite spectral triples and generalizations of the APS index theorem

is explained in [5] with reference to the extensive previous history of the matter.

Spectral flow is related to odd degree  $K$ -theory and the odd local index theorem in noncommutative geometry [20, 17]. For the purposes of using spectral flow to obtain information about  $K$ -theory it is enough to consider unbounded operators  $D$  with compact resolvent. However, previous studies of spectral flow formulae have restrictions on  $D$  such as theta summability (requiring the heat operator  $e^{-tD^2}$  to be trace class for all  $t > 0$ ). In this paper we will relax this condition to capture the more general situation where only compact resolvent is needed. By contrast the theory of the spectral shift function was formulated for Schrodinger operators associated to Euclidean spaces and in that setting the assumption of compact resolvent does not generally hold. Thus it is not surprising that the relationship between the Krein spectral shift function and spectral flow should have remained somewhat unexplored until recently.

In [3] we noticed that under hypotheses that guaranteed that both the Krein spectral shift function and spectral flow exist then they are essentially the same notion. This led us to the current investigation where we borrow some ideas from spectral flow theory as formulated in a spectral triple to extend the range of situations for which the Krein spectral shift function may be defined. We are then able to obtain new results on the spectral shift function and to prove, by a combination of the methods of [3] and [15, 16] that the spectral shift function gives a wide variety of analytic formulae for spectral flow. In fact it is the interaction between these previously disparate theories that makes these advances possible.

An additional very important feature of our analysis is its generality; we are able to study self adjoint operators that are affiliated to a general semifinite von Neumann algebra  $\mathcal{N}$  in contrast to the classical theory which deals with the special case where  $\mathcal{N}$  is the algebra of all bounded operators on a Hilbert space  $\mathcal{H}$ . (Note that there are examples of quantum mechanical Hamiltonians whose resolvent lies in a semifinite von Neumann algebra, see e.g. [14])

**Summary of results.** In order to keep this paper to a reasonable length we have omitted a detailed discussion of spectral flow in von Neumann algebras referring to the paper [5] for this. On the other hand we include in Section 1 preliminary results particularly those from [3] which may be less accessible to the reader. This latter paper is very important for the present investigation because it provides a calculus of functions of operators more effective than has been available in the past. The key technical advance that we exploit is the use of double operator integrals (DOI) not in its original form [8] but in the much more effective form discovered in [3]. In particular it is

this latter paper that develops a calculus for functions of operators affiliated to general semifinite von Neumann algebras. This calculus replaces the more elementary perturbation theory techniques in [15, 16] and is more user friendly than that of [27].

The main new results on the Krein spectral shift function are in Section 2. Here we show that by starting with self adjoint operators  $D_0$  with compact resolvent in a semifinite von Neumann algebra we are able to define the Krein spectral shift function for general bounded self adjoint perturbations  $V$ . Section 2 also exhibits some properties of the spectral shift function for operators satisfying these hypotheses mostly in the context of preparing the ground for the subsequent discussion of spectral flow. Specifically we observe that the spectral projections  $E_{(a,b)}^{D_0}$  and  $E_{(a,b)}^{D_0+V}$  of  $D_0$  and  $D_0 + V = D_1$  respectively are finite in  $\mathcal{N}$  for bounded intervals  $(a, b)$ . Using a fixed faithful normal semifinite trace  $\tau$  on  $\mathcal{N}$  we define (following [7]) the spectral shift measure for the pair  $D_0, D_1$ , by

$$\Xi_{D_1, D_0}(\Delta) = \int_0^1 \tau \left( V E_{\Delta}^{D_r} \right) dr,$$

where  $\Delta$  is a bounded Borel subset of the real line and  $D_r = D_0 + rV$ ,  $r \in [0, 1]$ . We consider this Birman-Solomyak formula (formula (16) in the text) for the spectral shift measure as fundamental. It is this formula which in our opinion must be taken as the definition of the generalized spectral shift function of a pair of operators, whenever this expression makes sense for that pair. We then prove that this measure is absolutely continuous with respect to Lebesgue measure and the resulting Radon-Nikodym derivative we define to be the spectral shift function  $\xi_{D_1, D_0}(\lambda)$ . This function can be related to the original spectral shift function of Krein which was introduced in a completely different fashion. Furthermore, under our assumptions, the spectral shift function  $\xi_{D_1, D_0}(\lambda)$  is in fact a function of a bounded variation and there is a canonical representative which is an everywhere defined function. We remark that in [33] B. Simon outlined (without proof) some similar results on the spectral shift function.

In Section 3 we present a series of new analytic formulae for spectral flow. These require the use of the spectral shift function and its properties derived in Section 2. The formulae we obtain subsume those of [16]. We are able to provide a variety of analytic formulae whenever  $D_0$  has compact resolvent and show how these formulae can be specialized when there are summability hypotheses imposed.

For illustrative purposes we describe two of our main theorems here. Suppose that there is a unitary operator  $u$  with  $D_1 = uD_0u^*$  and such that  $V = u[D_0, u^*] \in \mathcal{N}$ . That is, for a dense subalgebra of the  $C^*$ -algebra generated by  $u$  we have a semifinite spectral triple. Then spectral flow from  $D_0 - \lambda$  to  $D_1 - \lambda$  for any real  $\lambda$  is equal to  $\xi_{D_1, D_0}(\lambda)$ . We relate our results

to previous formulae for spectral flow via the following result. For any positive integrable function  $f$  on the real line with  $f(D_0 + rV)$  trace class for  $r \in [0, 1]$  and  $r \mapsto \|f(D_0 + rV)\|_1$  being 1-summable on  $[0, 1]$ , we find that spectral flow from  $D_0 - \lambda$  to  $D_1 - \lambda$  for any real  $\lambda$  is given by

$$\text{sf}(\lambda; D_0, D_1) = C^{-1} \int_0^1 \tau(Vf(D_r - \lambda)) \, dr,$$

where  $C = \int_{-\infty}^{\infty} f(x) \, dx$ . In fact in this restricted situation of unitarily equivalent endpoints we can prove that the spectral shift function is constant implying that spectral flow occurs uniformly past any point on the real line not just zero. We also obtain analogues of these results when the endpoint operators are not unitarily equivalent. Then the relation between spectral flow and the spectral shift function is modified by the addition of endpoint correction terms (in a fashion analogous to [16]). The key idea we exploit in this part of the paper is the observation [24] that spectral flow can be written as the integral of an exact one form on the affine space of bounded perturbations of a fixed unbounded operator  $D_0$ . It eventuates that there is a way to construct a large class of such one forms that will compute spectral flow when the only constraint on  $D_0$  is that it have compact resolvent. This advance is made possible by the understanding of the spectral shift measure afforded by Section 2. It then follows that the spectral shift function computes spectral flow with the caveat that there are simple correction terms arising from the zero eigenspace of each endpoint of the path.

One motivation for this investigation arises from [17] where the odd local index theorem in noncommutative geometry was deduced from the analytic formula for spectral flow in [16] via an intermediate formula in terms of a cyclic cocycle which was called the ‘resolvent cocycle’. The results and viewpoint of this paper may have other applications to noncommutative geometry besides spectral flow. Specifically we have in mind situations where we would like to avoid ‘summability constraints’ on  $D_0$ .

## 1. NOTATION AND PRELIMINARY RESULTS

**1.1. Notation.** We write  $\mathcal{H}$  for a fixed separable complex Hilbert space, and by  $\mathcal{N}$  we denote a semifinite von Neumann algebra acting on  $\mathcal{H}$  [34, V. 1. 21]. We use the usual notation  $D\eta\mathcal{N}$  for operators  $D$  affiliated with  $\mathcal{N}$  [34, IV. 5, Exercise 3]. We let  $\tau$  be a faithful normal semifinite trace on  $\mathcal{N}$  [34, V. 2. 1] and by  $L^1(\mathcal{N}, \tau)$  we denote the set of  $\tau$ -trace class operators affiliated with  $\mathcal{N}$  [34, V.2, p. 320]. Then we use the notation  $\mathcal{L}^1(\mathcal{N}, \tau) = L^1(\mathcal{N}, \tau) \cap \mathcal{N}$  for the (unitarily) invariant operator ideal [16, Appendix A.2] of all bounded  $\tau$ -trace class operators with norm  $\|\cdot\|_{1,\infty} = \|\cdot\| + \|\cdot\|_1$ , where  $\|\cdot\|$  is the usual operator norm and  $\|\cdot\|_1 := \tau(|\cdot|)$  [34, V.2, p. 320]. By  $\mathcal{B}(\mathbb{R})$  we denote the  $\sigma$ -algebra of all Borel subsets of  $\mathbb{R}$ . For a self-adjoint operator  $T$  let  $E_{\Delta}^T$  be the spectral projection of  $T$  corresponding to  $\Delta \in \mathcal{B}(\mathbb{R})$ , and let

$E_\lambda^T$  be the spectral projection of  $T$  corresponding to  $(-\infty, \lambda]$ . Write  $C_c^\infty(\mathbb{R})$  (respectively,  $C_c^\infty(\Omega)$ ) for the set of all compactly supported  $C^\infty$ -smooth functions on  $\mathbb{R}$  (respectively,  $\Omega \subseteq \mathbb{R}$ ), and  $B(\mathbb{R})$  (respectively,  $B_c(\mathbb{R})$ ) for set of all bounded Borel functions on  $\mathbb{R}$  (respectively, compactly supported bounded Borel functions on  $\mathbb{R}$ ).

If an operator  $T$  affiliated with  $\mathcal{N}$  is  $\tau$ -measurable [22, Definition 1.2], then the  $t$ -th generalized  $s$ -number  $\mu_t(T)$  of the operator  $T$  is defined by

$$\mu_t(T) = \inf \{ \|TE\| : E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - E) \leq t \}.$$

If  $E \in \mathcal{N}$  is a  $\tau$ -finite projection (i.e.  $\tau(E) < \infty$ ), then

$$(1) \quad \mu_t(E) = \chi_{[0, \tau(E))}(t), \quad t \geq 0.$$

An operator  $T \in \mathcal{N}$  is said to be  $\tau$ -compact iff  $\lim_{t \rightarrow \infty} \mu_t(T) = 0$ . The set of all  $\tau$ -compact operators from  $\mathcal{N}$  forms an ideal in  $\mathcal{N}$  which is denoted  $\mathcal{K}(\mathcal{N}, \tau)$ . The ideal  $\mathcal{K}(\mathcal{N}, \tau)$  coincides with the norm closure of the ideal in  $\mathcal{N}$  generated by  $\tau$ -finite projections.

Let  $R_z(D) = (z - D)^{-1}$  denote the resolvent of  $D$ . If  $D = D^* \eta \mathcal{N}$  and  $R_z(D) \in \mathcal{K}(\mathcal{N}, \tau)$  for some (and hence for all)  $z \in \mathbb{C} \setminus \mathbb{R}$ , then we say that  $D$  has  $\tau$ -compact resolvent.

An operator  $T \in \mathcal{N}$  is called  $\tau$ -Fredholm [29, Appendix B] if the projections  $[\ker T]$ ,  $[\ker T^*]$  are  $\tau$ -finite and there exists a  $\tau$ -finite projection  $E \in \mathcal{N}$  such that  $\text{ran}(1 - E) \subseteq \text{ran}(T)$ . For a  $\tau$ -Fredholm operator  $T$  one can define its  $\tau$ -index by  $\tau\text{-ind}(T) = \tau([\ker T]) - \tau([\ker T^*])$ .

**Definition 1.1.** Let  $\mathcal{N}$  be a semifinite von Neumann algebra and let  $\mathcal{E}, \mathcal{E}'$  be two invariant operator ideals over  $\mathcal{N}$ . We denote self-adjoint part of  $\mathcal{E}$  by  $\mathcal{E}_{sa}$ . Let  $D_0$  be a fixed self-adjoint operator affiliated with  $\mathcal{N}$ . A function  $f: D_0 + \mathcal{E}_{sa} \mapsto f(D_0) + \mathcal{E}'_{sa}$  is called affinely  $(\mathcal{E}, \mathcal{E}')$ -Fréchet differentiable at  $D \in D_0 + \mathcal{E}_{sa}$ , if there exists a (necessarily unique) bounded operator  $L: \mathcal{E}_{sa} \rightarrow \mathcal{E}'_{sa}$  such that the following equality holds

$$f(D + V) - f(D) = L(V) + o(\|V\|_{\mathcal{E}}), \quad V \in \mathcal{E}_{sa}.$$

If  $\mathcal{E} = \mathcal{E}'$  then we write  $L = D_{\mathcal{E}} f(D)$ , otherwise we write  $L = D_{\mathcal{E}, \mathcal{E}'} f(D)$ .

In our case the ideals  $\mathcal{E}$  and  $\mathcal{E}'$  will be  $\mathcal{N}$  or  $\mathcal{L}^1(\mathcal{N}, \tau)$ .

We constantly use some parameters for specific purposes. The parameter  $r$  will always be an operator path parameter, i.e. the letter  $r$  is used when we consider paths of operators such as  $D_r = D_0 + rV$ . Very rarely we need another path parameter which we denote by  $s$ . We do not use  $t$  as path parameter, since  $t$  is used for other purposes later in the paper. The letter  $\lambda$  is always used as a spectral parameter. If we need another spectral parameter we will use  $\mu$ .

We finish this subsection with the following elementary fact which we will use repeatedly.

**Lemma 1.2.** *If  $\Omega$  is an open interval in  $\mathbb{R}$  and if  $f \in C_c^k(\Omega)$ , then there exist functions  $f_1, f_2 \in C_c^k(\Omega)$  such that  $f_1, f_2$  are non-negative,  $f = f_1 - f_2$  and  $\sqrt{f_1}, \sqrt{f_2} \in C_c^k(\Omega)$ .*

*Proof.* Let  $[a, b]$  be a closed interval,  $[a, b] \subseteq \Omega$  and  $\text{supp}(f) \subseteq (a, b)$ . Take a non-negative  $C^\infty$ -function  $f_1 \geq f$  on  $[a, b]$  which vanishes at  $a$  and  $b$  in such a way that  $\sqrt{f_1}$  is  $C^\infty$ -smooth at  $a$  and  $b$ , and take  $f_2 = f_1 - f$ .  $\square$

**1.2. Self-adjoint operators with  $\tau$ -compact resolvent.** The remaining subsections in this Section are technical and dry. The reader may wish to browse this Section then move on to Sections 2 and 3 which contain the main results referring back to the technicalities of this Section when needed.

In this current subsection we collect some facts about operators with compact resolvent in a semifinite von Neumann algebra. We do not claim any great originality for these.

**Lemma 1.3.** *If  $D = D^*\eta\mathcal{N}$  has  $\tau$ -compact resolvent and  $V = V^* \in \mathcal{N}$ , then  $D + V$  also has  $\tau$ -compact resolvent.*

*Proof.* This follows from the equality  $R_z(D + V) = (R_z(D + V)V + 1)R_z(D)$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ .  $\square$

**Lemma 1.4.** *If  $D = D^*\eta\mathcal{N}$  has  $\tau$ -compact resolvent, then for all compact sets  $\Delta \subseteq \mathbb{R}$  the spectral projection  $E_\Delta^D$  is  $\tau$ -finite.*

*Proof.* If  $D$  has  $\tau$ -compact resolvent then the operator  $(1 + D^2)^{-1} = (D + i)^{-1}(D - i)^{-1}$  is  $\tau$ -compact. Since for every finite interval  $\Delta$  there exists a constant  $c > 0$ , not depending on  $D$  such that  $E_\Delta^D \leq c(1 + D^2)^{-1}$ , the projection  $E_\Delta^D$  is also  $\tau$ -compact, and hence  $\tau$ -finite.  $\square$

**Corollary 1.5.** *If  $D = D^*\eta\mathcal{N}$  has  $\tau$ -compact resolvent, then for all  $f \in B_c(\mathbb{R})$  the operator  $f(D)$  is  $\tau$ -trace class.*

*Proof.* There exists a finite segment  $\Delta \subseteq \mathbb{R}$  such that  $|f| \leq \text{const } \chi_\Delta$ , so that  $|f(D)| \leq \text{const } E_\Delta^D$ .  $\square$

**Lemma 1.6.** [15, Appendix B, Lemma 6] *If  $D_0$  is an unbounded self-adjoint operator,  $A$  is a bounded self-adjoint operator, and  $D = D_0 + A$  then*

$$(1 + D^2)^{-1} \leq f(\|A\|)(1 + D_0^2)^{-1},$$

where  $f(a) = 1 + \frac{1}{2}a^2 + \frac{1}{2}a\sqrt{a^2 + 4}$ .

**Lemma 1.7.** *Let  $D_0 = D_0^* \eta \mathcal{N}$  have  $\tau$ -compact resolvent, and let  $B_R = \{V = V^* \in \mathcal{N} : \|V\| \leq R\}$ . Then for any compact subset  $\Delta \subseteq \mathbb{R}$  the function*

$$V \in B_R \mapsto E_\Delta^{D_0+V}$$

*is  $\mathcal{L}^1(\mathcal{N}, \tau)$ -bounded.*

*Proof.* We have  $E_\Delta^{D_0+V} \leq c_0(1 + (D_0 + V)^2)^{-1}$  for some constant  $c_0 = c_0(\Delta) > 0$  and for every  $V = V^* \in \mathcal{N}$ . Now, by Lemma 1.6 there exists a constant  $c_1 = c_1(R) > 0$ , such that for all  $V \in B_R$

$$(1 + (D_0 + V)^2)^{-1} \leq c_1(1 + D_0^2)^{-1}.$$

Hence, since  $D_0$  has  $\tau$ -compact resolvent, all projections  $E_\Delta^{D_0+V}$ ,  $V \in B_R$ , are bounded from above by a single  $\tau$ -compact operator  $T = c_0 c_1(1 + D_0^2)^{-1}$ . This means, that for  $t > 0$

$$\mu_t(E_\Delta^{D_0+V}) \leq \mu_t(T).$$

Further, by (1)  $\mu_t(E_\Delta^{D_0+V}) = \chi_{[0, \tau(E_\Delta^{D_0+V}))}(t)$  and there exists  $t_0 > 0$  such that  $\mu_{t_0}(T) \leq 1$ . This implies that for all  $V \in B_R$ ,  $\tau(E_\Delta^{D_0+V}) \leq t_0$ .  $\square$

**Corollary 1.8.** *If  $D_0 = D_0^* \eta \mathcal{N}$  has  $\tau$ -compact resolvent, then for any function  $f \in B_c(\mathbb{R})$  the function  $V \in B_R \mapsto \|f(D_0 + V)\|_{1, \infty}$  is bounded.*

**Corollary 1.9.** *Let  $D_0 = D_0^* \eta \mathcal{N}$  have  $\tau$ -compact resolvent,  $r = (r_1, \dots, r_m) \in [0, 1]^m$ ,  $V_1, \dots, V_m \in \mathcal{N}_{sa}$  and set  $D_r = D_0 + r_1 V_1 + \dots + r_m V_m$ . Then*

- (i) *for any compact subset  $\Delta \subseteq \mathbb{R}$  the function  $r \in [0, 1]^m \mapsto \|E_\Delta^{D_r}\|_1$  is bounded;*
- (ii) *for any function  $f \in B_c(\mathbb{R})$  the function  $r \in [0, 1]^m \mapsto \|f(D_r)\|_1$  is bounded.*

An elementary proof of the following lemma can also be found in [15].

**Lemma 1.10.** *If  $D_0 = D_0^* \eta \mathcal{N}$  and if  $V = V^* \in \mathcal{N}$ , then for any  $t \in \mathbb{R}$ ,  $e^{it(D_0+V)}$  converges in  $\|\cdot\|$ -norm to  $e^{itD_0}$  when  $\|V\| \rightarrow 0$ .*

*Proof.* Follows directly from Duhamel's formula

$$e^{it(D_0+V)} - e^{itD_0} = \int_0^t e^{i(t-u)(D_0+V)} iV e^{iuD_0} du.$$

$\square$

If  $(S, \nu)$  is a finite measure space then we say that a function  $f: S \rightarrow \mathcal{B}(\mathcal{H})$  is measurable, if for any  $\eta \in \mathcal{H}$  the functions  $f(\cdot)\eta, f(\cdot)^*\eta: S \rightarrow \mathcal{H}$  are



Bochner measurable. We define the integral of a measurable function by the formula

$$\int_S f(\sigma) d\nu(\sigma)\eta = \int_S f(\sigma)\eta d\nu(\sigma), \quad \eta \in \mathcal{H},$$

and call it an  $so^*$ -integral. If  $f: S \rightarrow \mathcal{L}^1(\mathcal{N}, \tau)$  is an  $\mathcal{L}^1$ -bounded function then it is measurable if and only if there exists a sequence of simple (finitely-valued) functions  $f_n: S \rightarrow \mathcal{L}^1(\mathcal{N}, \tau)$  such that  $f_n(\sigma) \rightarrow f(\sigma)$  in the  $so^*$ -topology [3, Section 3].

**Lemma 1.11.** [3, Lemma 3.10] *If a measurable function  $f: S \rightarrow \mathcal{L}^1(\mathcal{N}, \tau)$  is uniformly  $\mathcal{L}^1(\mathcal{N}, \tau)$ -bounded then  $\int_S f(\sigma) d\nu(\sigma) \in \mathcal{L}^1(\mathcal{N}, \tau)$  and*

$$\tau\left(\int_S f(\sigma) d\nu(\sigma)\right) = \int_S \tau(f(\sigma)) d\nu(\sigma).$$

**1.3. Difference quotients and double operator integrals.** Originally, multiple operator integrals of the form

$$(2) \quad T_\varphi^{D_0, \dots, D_n}(V_1, \dots, V_n) = \int_{\mathbb{R}^{n+1}} \varphi(\lambda_0, \dots, \lambda_n) dE_{\lambda_0}^{D_0} V_1 dE_{\lambda_1}^{D_1} V_2 \dots V_n dE_{\lambda_n}^{D_n},$$

where  $D_0, \dots, D_n$  are self-adjoint operators and  $V_1, \dots, V_n$  are bounded operators, were defined as repeated operator integrals [21] or as spectral integrals [8, 9, 10] in the case of double operator integrals. It was noted in [3] (see also [28]) that one can give another definition of multiple operator integrals (Definition 1.15). The idea of the new definition is that for functions  $\varphi(\lambda_0, \dots, \lambda_n)$  of the form  $\varphi(\lambda_0, \dots, \lambda_n) = \alpha_0(\lambda_0)\alpha_1(\lambda_1)\dots\alpha_n(\lambda_n)$  the expression (2) can be interpreted as  $\alpha_0(D_0)V_1\alpha_1(D_1)V_2\dots V_n\alpha_n(D_n)$ . Hence, for functions  $\varphi$  of the form (7) one can try to define the multiple operator integral by the formula (8). Having proved the correctness of this definition (Proposition 1.16), we will be able to work with the usual operator integrals of the form (8), provided that one has a representation of the form of equation (7) for the function  $\varphi$ . Actually, one has considerable freedom of choice of this representation, and one should try to find that representation of  $\varphi$  which is most suitable for the purpose at hand.

In our case the function  $\varphi(\lambda_0, \lambda_1)$  is the difference quotient

$$(3) \quad f^{[1]}(\lambda_0, \lambda_1) := \frac{f(\lambda_0) - f(\lambda_1)}{\lambda_0 - \lambda_1}$$

of some function  $f$ . We denote by  $C^{n,+}(\mathbb{R})$  the set of functions  $f \in C^n(\mathbb{R})$ , such that the  $j$ -th derivative  $f^{(j)}$ ,  $j = 1, \dots, n$ , belongs to the space  $\mathcal{F}^{-1}(L^1(\mathbb{R}))$ , where  $\mathcal{F}(f)$  is the Fourier transform of  $f$ . We emphasize that the Fourier transform of a function  $f \in C^{n,+}(\mathbb{R})$  need not to belong to  $L^1(\mathbb{R})$  (see Lemma 1.12(ii)). Let

$$\Pi := \{(s_0, s_1) \in \mathbb{R}^2: |s_1| \leq |s_0|, \text{sign}(s_0) = \text{sign}(s_1)\},$$

and

$$(4) \quad d\nu_f(s_0, s_1) := \operatorname{sgn}(s_0) \frac{i}{\sqrt{2\pi}} \mathcal{F}(f)(s_0) ds_0 ds_1.$$

If  $f \in C^{1,+}(\mathbb{R})$ , then it is not difficult to see that  $(\Pi, \nu_f)$  is a finite measure space [3, Lemma 2.1], so that  $(\Pi, \nu_f)$  can be used for the construction of the double operator integral. The following Birman-Solomyak or BS-representation of  $f^{[1]}(\lambda_0, \lambda_1)$  [3, Lemma 2.2]

$$(5) \quad f^{[1]}(\lambda_0, \lambda_1) = \int_{\Pi} \alpha(\lambda_0, \sigma) \beta(\lambda_1, \sigma) d\nu_f(\sigma),$$

where  $\sigma = (s_0, s_1)$ ,  $\alpha(\lambda_0, \sigma) = e^{i(s_0 - s_1)\lambda_0}$  and  $\beta(\lambda_1, \sigma) = e^{is_1\lambda_1}$ , is not suitable for our present purposes. Lemma 1.14 provides a modification of this BS-representation for  $f^{[1]}$  with which we will constantly work.

We include the proof of the following fact for completeness.

**Lemma 1.12.** (i) If  $f \in C_c^1(\mathbb{R})$ , then  $\hat{f} \in L^1(\mathbb{R})$ .  
(ii) the function  $\varphi(x) = \frac{x}{\sqrt{1+x^2}}$  belongs to  $C^{2,+}(\mathbb{R})$ .

*Proof.* (i) Following the proof of [13, Corollary 3.2.33], we have

$$\begin{aligned} \int_{\mathbb{R}} |\hat{f}|(\xi) d\xi &= \int_{\mathbb{R}} |\xi + i|^{-1} |\xi + i| |\hat{f}|(\xi) d\xi \\ &\leq \left( \int_{\mathbb{R}} |\xi + i|^{-2} d\xi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} |\xi + i|^2 |\hat{f}|^2(\xi) d\xi \right)^{\frac{1}{2}} \\ &= \operatorname{const} \left( \int_{\mathbb{R}} \left| \frac{df(x)}{dx} + f(x) \right|^2 dx \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

(ii) The proof is similar to that of (i). □

Applying part (i) of this lemma to the first  $n$  derivatives of a function  $f$  from  $C_c^{n+1}(\mathbb{R})$ , we obtain

**Corollary 1.13.**  $C_c^{n+1}(\mathbb{R}) \subseteq C^{n,+}(\mathbb{R})$ ,  $n = 1, 2, \dots$

The following lemma provides a BS-representation for  $f^{[1]}$ ,  $0 \leq f \in C_c^2(\mathbb{R})$ , which will be used throughout this paper.

**Lemma 1.14.** Let  $f \in C_c^2(\mathbb{R})$  be a non-negative function such that  $g := \sqrt{f} \in C_c^2(\mathbb{R})$ . If  $\Omega \supseteq \operatorname{supp}(f)$ , then

$$f^{[1]}(\lambda_0, \lambda_1) = \int_{\Pi} (\alpha_1(\lambda_0, \sigma) \beta_1(\lambda_1, \sigma) + \alpha_2(\lambda_0, \sigma) \beta_2(\lambda_1, \sigma)) d\nu_g(\sigma),$$

where  $\sigma = (s_0, s_1)$  and

$$(6) \quad \begin{aligned} \alpha_1(\lambda_0, \sigma) &= e^{i(s_0-s_1)\lambda_0} g(\lambda_0), & \beta_1(\lambda_1, \sigma) &= e^{is_1\lambda_1}, \\ \alpha_2(\lambda_0, \sigma) &= e^{i(s_0-s_1)\lambda_0}, & \beta_2(\lambda_1, \sigma) &= e^{is_1\lambda_1} g(\lambda_1), \end{aligned}$$

so that  $\alpha_1(\cdot, \sigma), \beta_2(\cdot, \sigma) \in C_c^2(\Omega)$  for all  $\sigma \in \Pi$ , and  $|\alpha_1(\cdot)|, |\beta_2(\cdot)| \leq \|g\|_\infty$ , while  $\alpha_2(\cdot, \sigma), \beta_1(\cdot, \sigma) \in C^\infty(\mathbb{R})$  for all  $\sigma \in \Pi$ , and  $|\alpha_2(\cdot)|, |\beta_1(\cdot)| \leq 1$ .

*Proof.* The assumption  $g \in C_c^2(\mathbb{R})$  implies that  $g \in C^{1,+}(\mathbb{R})$  (see Corollary 1.13). Now,

$$\begin{aligned} f^{[1]}(\lambda_0, \lambda_1) &= \frac{g^2(\lambda_0) - g^2(\lambda_1)}{\lambda_0 - \lambda_1} \\ &= \frac{g(\lambda_0) - g(\lambda_1)}{\lambda_0 - \lambda_1} (g(\lambda_0) + g(\lambda_1)) = g^{[1]}(\lambda_0, \lambda_1) (g(\lambda_0) + g(\lambda_1)). \end{aligned}$$

Hence, using (5), we have

$$f^{[1]}(\lambda_0, \lambda_1) = \int_{\Pi} \left( \alpha(\lambda_0, \sigma) g(\lambda_0) \beta(\lambda_1, \sigma) + \alpha(\lambda_0, \sigma) g(\lambda_1) \beta(\lambda_1, \sigma) \right) d\nu_g(\sigma).$$

If we set  $\alpha_1(\lambda, \sigma) = \alpha(\lambda, \sigma) g(\lambda)$ ,  $\beta_1(\lambda, \sigma) = \beta(\lambda, \sigma)$ ,  $\alpha_2(\lambda, \sigma) = \alpha(\lambda, \sigma)$  and  $\beta_2(\lambda, \sigma) = g(\lambda) \beta(\lambda, \sigma)$ , then we see that all the conditions of the Lemma are fulfilled.  $\square$

The next step is to recall the definition of the multiple operator integral as it was given in [3]. Let  $\varphi \in B(\mathbb{R}^{n+1})$  be a bounded Borel function on  $\mathbb{R}^{n+1}$  which admits a representation of the form

$$(7) \quad \varphi(\lambda_0, \lambda_1, \dots, \lambda_n) = \int_S \alpha_0(\lambda_0, \sigma) \dots \alpha_n(\lambda_n, \sigma) d\nu(\sigma),$$

where  $(S, \nu)$  is a finite measure space and  $\alpha_0, \dots, \alpha_n$  are bounded Borel functions on  $\mathbb{R} \times S$ .

**Definition 1.15.** For arbitrary self-adjoint operators  $D_0, \dots, D_n$  on the separable Hilbert space  $\mathcal{H}$ , bounded operators  $V_1, \dots, V_n$  on  $\mathcal{H}$  and any function  $\varphi \in B(\mathbb{R}^{n+1})$  which admits a representation given by (7), the multiple operator integral  $T_\varphi^{D_0, \dots, D_n}(V_1, \dots, V_n)$  is defined as

$$(8) \quad T_\varphi^{D_0, \dots, D_n}(V_1, \dots, V_n) := \int_S \alpha_0(D_0, \sigma) V_1 \dots V_n \alpha_n(D_n, \sigma) d\nu(\sigma),$$

where the integral is taken in the  $so^*$ -topology.

**Proposition 1.16.** [3, Lemma 4.3] The multiple operator integral in Definition 1.15 is well-defined in the sense that it does not depend on the representation (7) of  $\varphi$ .

The following lemma is a corollary of Lemma 1.14 and the definition of the multiple operator integral.

**Lemma 1.17.** *If  $D_0 = D_0^* \eta \mathcal{N}$ , if  $D_1 = D_0 + V$ ,  $V = V^* \in \mathcal{N}$  and if  $f \in C_c^2(\mathbb{R})$  is a non-negative function, such that  $g := \sqrt{f} \in C_c^2(\mathbb{R})$ , then*

$$T_{f[1]}^{D_1, D_0}(V) = \int_{\Pi} (\alpha_1(D_1, \sigma) V \beta_1(D_0, \sigma) + \alpha_2(D_1, \sigma) V \beta_2(D_0, \sigma)) \, d\nu_g(\sigma),$$

where  $\alpha_1, \beta_1, \alpha_2, \beta_2$  are given by (6).

We need the following weaker version of [3, Theorem 5.3]. See also [10].

**Proposition 1.18.** [3, Theorem 5.3] *Let  $\mathcal{N}$  be a von Neumann algebra. Suppose that  $D_0 = D_0^*$  is affiliated with  $\mathcal{N}$ , that  $V \in \mathcal{N}$  is self-adjoint and set  $D_1 = D_0 + V$ .*

(i) *If  $f \in C^{1,+}(\mathbb{R})$ , then*

$$f(D_1) - f(D_0) = T_{f[1]}^{D_1, D_0}(V).$$

(ii) *If  $f \in C^{2,+}(\mathbb{R})$ , then the function  $f: D_0 + \mathcal{N}_{sa} \mapsto f(D_0) + \mathcal{N}_{sa}$  is affinely  $(\mathcal{N}, \mathcal{N})$ -Fréchet differentiable, the equality  $D_{\mathcal{N}} f(D) = T_{f[1]}^{D, D}$  holds and  $D_{\mathcal{N}} f(D)$  is  $\|\cdot\|$ -continuous, where  $D_{\mathcal{N}}$  is to be understood in the sense of Definition 1.1.*

**1.4. Some continuity and differentiability properties of operator functions.** We are going to consider spectral flow along ‘continuous’ paths of unbounded Fredholm operators. We will make precise what we mean by continuity in this setting later. However our formulae require more than just continuity. They require us to be able to take derivatives with the respect to the path parameter. For this to be feasible we need the full force of the double operator integral formalism. We present the results we will need as a sequence of lemmas.

**Lemma 1.19.** *Let  $(S, \nu)$  be a finite measure space and let  $f: S \rightarrow \mathcal{L}^1(\mathcal{N}, \tau)$  be a  $\mathcal{L}^1(\mathcal{N}, \tau)$ -bounded  $so^*$ -measurable function. Then*

$$\left\| \int f(\sigma) \, d\nu(\sigma) \right\|_{\mathcal{L}^1} \leq \int \|f(\sigma)\|_{\mathcal{L}^1} \, d|\nu|(\sigma).$$

*Proof.* By definition for any  $\eta \in \mathcal{H}$  the function  $\sigma \mapsto f(\sigma)\eta$  is Bochner measurable. Hence, the function  $\sigma \mapsto \|f(\sigma)\| = \sup_{\eta \in \mathcal{H}: \|\eta\| \leq 1} \|f(\sigma)\eta\|$  is also measurable. Similarly, since the function  $\sigma \mapsto \tau(f(\sigma)B)$  is measurable, the function  $\sigma \mapsto \|f(\sigma)\|_1 = \sup_{B \in \mathcal{N}: \|B\| \leq 1} |\tau(f(\sigma)B)|$  is also measurable. Hence, the right hand side of the last equality is well defined.

For  $\eta \in \mathcal{H}$  with  $\|\eta\| \leq 1$  we have by definition of  $so^*$ -integral (see [3, (2)])

$$(9) \quad \left\| \int f(\sigma) d\nu(\sigma) \eta \right\| = \left\| \int f(\sigma) \eta d\nu(\sigma) \right\| \\ \leq \int \|f(\sigma) \eta\| d|\nu|(\sigma) \leq \int \|f(\sigma)\| d|\nu|(\sigma).$$

Hence, the inequality is true for  $\|\cdot\|$ -norm. Since  $\|A\|_1 = \sup_{B \in \mathcal{N}: \|B\| \leq 1} |\tau(AB)|$ , we have

$$(10) \quad \left\| \int f(\sigma) d\nu(\sigma) \right\|_1 = \sup_{B \in \mathcal{N}: \|B\| \leq 1} \left| \tau \left( \int f(\sigma) d\nu(\sigma) B \right) \right| \\ = \sup_{B \in \mathcal{N}: \|B\| \leq 1} \left| \int \tau(f(\sigma) B) d\nu(\sigma) \right| \\ \leq \sup_{B \in \mathcal{N}: \|B\| \leq 1} \int |\tau(f(\sigma) B)| d\nu(\sigma) \leq \int \|f(\sigma)\|_1 d|\nu|(\sigma),$$

where the second equality follows from the definition of  $so^*$ -integral and Lemma 1.11. Combining (9) and (10) completes the proof.  $\square$

In the sequel we will constantly need to take functions of a path of operators. We thus need the following continuity result. For the definition of  $B_R$  see Lemma 1.7.

**Proposition 1.20.** *If  $D_0 = D_0^* \eta \mathcal{N}$  has  $\tau$ -compact resolvent and if  $f \in C_c^2(\mathbb{R})$  then the operator-valued function  $A: V \in B_R \mapsto f(D_0 + V)$  takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)$  and is  $\mathcal{L}^1(\mathcal{N}, \tau)$ -continuous.*

*Proof.* That  $A(\cdot)$  takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)$  follows from Lemma 1.3 and Corollary 1.5. By Lemma 1.2 it is enough to prove continuity for a non-negative function  $f$  with  $g = \sqrt{f} \in C_c^2(\mathbb{R})$ . By Proposition 1.18(i) and Lemma 1.17 we have

$$f(D_0 + V) - f(D_0) = T_{f^{[1]}}^{D_0+V, D_0}(V) \\ = \int_{\Pi} (\alpha_1(D_0 + V, \sigma) V \beta_1(D_0, \sigma) + \alpha_2(D_0 + V, \sigma) V \beta_2(D_0, \sigma)) d\nu_g(\sigma).$$

Hence, by Lemma 1.19, we have

$$\|f(D_0 + V) - f(D_0)\|_{1, \infty} \leq \int_{\Pi} [\|\alpha_1(D_0 + V, \sigma)\|_{1, \infty} \|V\| \|\beta_1(D_0, \sigma)\| \\ + \|\alpha_2(D_0 + V, \sigma)\| \|V\| \|\beta_2(D_0, \sigma)\|_{1, \infty}] d|\nu_g|(\sigma) \\ \leq \int_{\Pi} (\|g(D_0 + V)\|_{1, \infty} \|V\| + \|V\| \|g(D_0)\|_{1, \infty}) d|\nu_g|(\sigma) \\ \leq |\nu_g|(\Pi) \|V\| (\|g(D_0 + V)\|_{1, \infty} + \|g(D_0)\|_{1, \infty}).$$

Now, Corollary 1.8 applied to  $g$  completes the proof.  $\square$

**Corollary 1.21.** *If  $D_0 = D_0^* \eta \mathcal{N}$  has  $\tau$ -compact resolvent,  $r = (r_1, \dots, r_m) \in [a, b]^m$ , if  $V_1, \dots, V_m \in \mathcal{N}_{sa}$  and if  $D_r = D_0 + rV = D_0 + r_1 V_1 + \dots + r_m V_m$ , then for any function  $f \in C_c^2(\mathbb{R})$  the operator-valued function  $A: r \in [a, b]^m \mapsto f(D_0 + rV)$  takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)$  and is  $\mathcal{L}^1(\mathcal{N}, \tau)$ -continuous.*

Next we prove the main lemmas of this Section. There are several matters to establish. First we want to be able to differentiate, with respect to the path parameter, certain functions of paths of operators. Then we need to determine formulae for the derivatives and the continuity properties of the derivatives with respect to the path parameter.

**Lemma 1.22.** *If  $D_1$  and  $D_2$  are two self-adjoint operators with  $\tau$ -compact resolvent affiliated with semifinite von Neumann algebra  $\mathcal{N}$ , if  $X \in \mathcal{N}_{sa}$  and if  $f \in C_c^3(\mathbb{R})$  then  $T_{f^{[1]}}^{D_1, D_2}(X)$  depends  $\mathcal{L}^1$ -continuously on  $\|\cdot\|$  perturbations of  $D_1$  and  $D_2$ .*

*Proof.* As usual, we can assume that  $f$  is non-negative and its square root  $g = \sqrt{f}$  is  $C^3$ -smooth.

Let  $Y_1, Y_2 \in \mathcal{N}_{sa}$ . Then by Lemma 1.17

$$\begin{aligned} & T_{f^{[1]}}^{D_1+Y_1, D_2+Y_2}(X) - T_{f^{[1]}}^{D_1, D_2}(X) \\ &= \int_{\Pi} [\alpha_1(D_1 + Y_1, \sigma) X \beta_1(D_2 + Y_2, \sigma) + \alpha_2(D_1 + Y_1, \sigma) X \beta_2(D_2 + Y_2, \sigma) \\ &\quad - \alpha_1(D_1, \sigma) X \beta_1(D_2, \sigma) - \alpha_2(D_1, \sigma) X \beta_2(D_2, \sigma)] d\nu_g(\sigma) \\ &= \int_{\Pi} \left( [\alpha_1(D_1 + Y_1, \sigma) - \alpha_1(D_1, \sigma)] X \beta_1(D_2 + Y_2, \sigma) \right. \\ &\quad \left. + \alpha_1(D_1, \sigma) X [\beta_1(D_2 + Y_2, \sigma) - \beta_1(D_2, \sigma)] \right. \\ &\quad \left. + [\alpha_2(D_1 + Y_1, \sigma) - \alpha_2(D_1, \sigma)] X \beta_2(D_2 + Y_2, \sigma) \right. \\ &\quad \left. + \alpha_2(D_1, \sigma) X [\beta_2(D_2 + Y_2, \sigma) - \beta_2(D_2, \sigma)] \right) d\nu_g(\sigma). \end{aligned}$$

For every fixed  $\sigma \in \Pi$  by Lemma 1.10 the norms  $\|\beta_1(D_2 + Y_2, \sigma) - \beta_1(D_2, \sigma)\|$  and  $\|\alpha_2(D_1 + Y_1, \sigma) - \alpha_2(D_1, \sigma)\|$  converge to zero when  $\|Y_1\|, \|Y_2\| \rightarrow 0$ , and by Corollary 1.8 the  $\mathcal{L}^1$ -norms of  $\alpha_1(D_1, \sigma)$  and  $\beta_2(D_2 + Y_2, \sigma)$  are bounded when  $\|Y_1\|, \|Y_2\| \rightarrow 0$ . Hence, for every fixed  $\sigma \in \Pi$  the  $\mathcal{L}^1$ -norms of the second and third summands in the last integral converge to zero when  $\|Y_1\|, \|Y_2\| \rightarrow 0$ .

Now we are going to show that the same is true for the first and fourth summands. It is enough to prove that for every fixed  $\sigma \in \Pi$ , for example,

$\|\alpha_1(D_1 + Y_1, \sigma) - \alpha_1(D_1, \sigma)\|_{\mathcal{L}^1}$  tends to zero. We have

$$\begin{aligned}
& \|\alpha_1(D_1 + Y_1, \sigma) - \alpha_1(D_1, \sigma)\|_{\mathcal{L}^1} \\
&= \left\| e^{i(s_0-s_1)(D_1+Y_1)} g(D_1 + Y_1) - e^{i(s_0-s_1)D_1} g(D_1) \right\|_{\mathcal{L}^1} \\
&\leq \left\| \left( e^{i(s_0-s_1)(D_1+Y_1)} - e^{i(s_0-s_1)D_1} \right) g(D_1 + Y_1) \right\|_{\mathcal{L}^1} \\
&\quad + \left\| e^{i(s_0-s_1)D_1} (g(D_1 + Y_1) - g(D_1)) \right\|_{\mathcal{L}^1} \\
&\leq \left\| \left( e^{i(s_0-s_1)(D_1+Y_1)} - e^{i(s_0-s_1)D_1} \right) \right\| \|g(D_1 + Y_1)\|_{\mathcal{L}^1} \\
&\quad + \|g(D_1 + Y_1) - g(D_1)\|_{\mathcal{L}^1}.
\end{aligned}$$

It follows from Lemma 1.10 that the first summand converges to zero when  $s_0, s_1$  are fixed and  $\|Y_1\| \rightarrow 0$ , and it follows from Proposition 1.20 that the second summand also converges to zero.

Since by Corollary 1.8 the trace norm of the expression under the last integral is uniformly  $\mathcal{L}^1(\mathcal{N}, \tau)$ -bounded with respect to  $\sigma \in \Pi$ , it follows from Lemma 1.19 that

$$\left\| T_{f^{[1]}}^{D_1+Y_1, D_2+Y_2}(X) - T_{f^{[1]}}^{D_1, D_2}(X) \right\|_{\mathcal{L}^1} \rightarrow 0,$$

when  $\|Y_1\|, \|Y_2\| \rightarrow 0$ .  $\square$

The following theorem is a version of a well-known Daletskii-S. G. Kreĭn formula [21]. We would like to give a heuristic argument explaining the formula (11). In the resolvent expansion series

$$\frac{1}{z - H - V} = \frac{1}{z - H} + \frac{1}{z - H} V \frac{1}{z - H} + \frac{1}{z - H} V \frac{1}{z - H} V \frac{1}{z - H} + \dots$$

the second summand is a double operator integral  $T_\varphi^{H,H}(V)$  with

$$\varphi(\lambda, \mu) = \frac{1}{z - \lambda} \cdot \frac{1}{z - \mu} = \frac{1}{\lambda - \mu} \left( \frac{1}{z - \lambda} - \frac{1}{z - \mu} \right).$$

If  $H$  and  $H + V$  are bounded operators and  $f$  is a function analytic in a neighbourhood of the union of the spectra of  $H$  and  $H + V$  then the Cauchy integral implies

$$f(H + V) = f(H) + T_{f^{[1]}}^{H,H}(V) + \text{terms of second order.}$$

**Theorem 1.23.** *If the von Neumann algebra  $\mathcal{N}$  is semifinite,  $D_0 = D_0^* \eta \mathcal{N}$  has  $\tau$ -compact resolvent and  $f \in C_c^3(\mathbb{R})$  then the function  $f: D \in D_0 + \mathcal{N}_{sa} \mapsto f(D) \in \mathcal{N}_{sa}$  takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)_{sa}$ . Moreover, it is affinely  $(\mathcal{N}, \mathcal{L}^1)$ -Fréchet differentiable, the equality*

$$(11) \quad D_{\mathcal{N}, \mathcal{L}^1} f(D) = T_{f^{[1]}}^{D,D}$$

holds, and  $D_{\mathcal{N}, \mathcal{L}^1} f(D)$  is  $(\mathcal{N}, \mathcal{L}^1)$ -continuous, so that

$$(12) \quad f(D_b) - f(D_a) = \int_a^b T_{f^{[1]}}^{D_r, D_r}(V) dr,$$

where  $V \in \mathcal{N}_{sa}$ ,  $D_r = D_0 + rV$  and the integral converges in  $\mathcal{L}^1(\mathcal{N}, \tau)$ -norm.

*Proof.* We have by Proposition 1.18(i) and Lemma 1.17

$$\begin{aligned} f(D_1) - f(D_0) &= T_{f^{[1]}}^{D_1, D_0}(V) \\ &= \int_{\Pi} (\alpha_1(D_1, \sigma)V\beta_1(D_0, \sigma) + \alpha_2(D_1, \sigma)V\beta_2(D_0, \sigma)) d\nu_g(\sigma), \\ &= \int_{\Pi} (\alpha_1(D_0, \sigma)V\beta_1(D_0, \sigma) + \alpha_2(D_0, \sigma)V\beta_2(D_0, \sigma)) d\nu_g(\sigma) \\ &\quad + \int_{\Pi} [\alpha_1(D_1, \sigma) - \alpha_1(D_0, \sigma)]V\beta_1(D_0, \sigma) d\nu_g(\sigma) \\ &\quad + \int_{\Pi} [\alpha_2(D_1, \sigma) - \alpha_2(D_0, \sigma)]V\beta_2(D_0, \sigma) d\nu_g(\sigma) \\ &= T_{f^{[1]}}^{D_0, D_0}(V) + (II) + (III). \end{aligned}$$

Since  $\alpha_2$  is just an exponent and since  $g \in C^{2,+}(\mathbb{R})$  that  $\|(III)\|_{\mathcal{L}^1} = O(\|V\|^2)$  can be shown by Duhamel's formula. The argument is as in the proof of [3, Theorem 5.5]. So, it is left to show that  $\|(II)\|_{\mathcal{L}^1}$  is  $o(\|V\|)$ . By Lemma 1.19 we have

$$\begin{aligned} \|(II)\|_{\mathcal{L}^1} &= \left\| \int_{\Pi} [\alpha_1(D_1, \sigma) - \alpha_1(D_0, \sigma)]V\beta_1(D_0, \sigma) d\nu_g(\sigma) \right\|_{\mathcal{L}^1} \\ &\leq \int_{\Pi} \|\alpha_1(D_1, \sigma) - \alpha_1(D_0, \sigma)\|_{\mathcal{L}^1} \|V\| \|\beta_1(D_0, \sigma)\| d\nu_g(\sigma) \\ &= \|V\| \int_{\Pi} \|\alpha_1(D_1, \sigma) - \alpha_1(D_0, \sigma)\|_{\mathcal{L}^1} d\nu_g(\sigma). \end{aligned}$$

Now, it follows from  $\alpha_1(\cdot, \sigma) \in C_c^2(\mathbb{R})$  (see (6)) and Proposition 1.20 that  $\|\alpha_1(D_1, \sigma) - \alpha_1(D_0, \sigma)\|_{\mathcal{L}^1} \rightarrow 0$ ,  $\sigma \in \Pi$ , so that by the Lebesgue dominated convergence theorem we conclude that the last integral converges to 0, and hence  $\|(II)\|_{\mathcal{L}^1} = o(\|V\|)$ .

Finally, that  $D_{\mathcal{N}, \mathcal{L}^1} f(D)$  is  $(\mathcal{N}, \mathcal{L}^1)$ -continuous follows from Lemma 1.22.  $\square$

**1.5. A class  $\mathcal{F}^{a,b}(\mathcal{N}, \tau)$  of  $\tau$ -Fredholm operators.** Our technique for handling spectral flow of paths of unbounded operators is to map them into the space of bounded operators using a particular function. We thus need to discuss some continuity properties of paths of bounded  $\tau$ -Fredholm operators, analogous to those we described in the unbounded case.



Let  $a < b$  be two real numbers. Let  $\mathcal{F}^{a,b}(\mathcal{N}, \tau)$  be the set of bounded self-adjoint  $\tau$ -Fredholm operators  $F \in \mathcal{N}$  such that  $(F-a)(F-b) \in \mathcal{K}(\mathcal{N}, \tau)$ . For  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$  let  $\mathcal{A}_{F_0} = F_0 + \mathcal{K}(\mathcal{N}, \tau)_{sa}$  be the affine space of  $\tau$ -compact self-adjoint perturbations of  $F_0$ .

**Lemma 1.24.** *If  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$  then*

$$\mathcal{A}_{F_0} \subseteq \mathcal{F}^{a,b}(\mathcal{N}, \tau).$$

*Proof.* If  $K \in \mathcal{K}(\mathcal{N}, \tau)_{sa}$  then  $(F_0 + K - a)(F_0 + K - b) = (F_0 - a)(F_0 - b) + (F_0 - a)K + K(F_0 + K - b) \in \mathcal{K}(\mathcal{N}, \tau)$ .  $\square$

**Lemma 1.25.** *If  $F \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$  and  $h \in B_c(a, b)$  then  $h(F) \in \mathcal{L}^1(\mathcal{N}, \tau)$ .*

*Proof.* The proof is similar to the proof of Lemma 1.4. For any compact subset  $\Delta$  of  $(a, b)$  there exists a constant  $c_0 > 0$  such that  $\chi_\Delta(x) \leq c_0 \chi_{[a,b]}(x)(b-x)(x-a)$ , so that

$$(13) \quad E_\Delta^F \leq c_0 E_{[a,b]}^F (b-F)(F-a).$$

Since  $(b-F)(F-a) \in \mathcal{K}(\mathcal{N}, \tau)$ , it follows that  $E_\Delta^F \in \mathcal{K}(\mathcal{N}, \tau)$  and hence  $E_\Delta^F$  is  $\tau$ -finite. Now, for any  $h \in B_c(a, b)$  there exists a compact subset  $\Delta$  of  $(a, b)$  and a constant  $c_1$  such that  $|h| \leq c_1 \chi_\Delta$ , so that  $|h(F)| \leq c_1 E_\Delta^F$  and hence  $h(F) \in \mathcal{L}^1(\mathcal{N}, \tau)$ .  $\square$

**Lemma 1.26.** *If  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ ,  $K = K^* \in \mathcal{K}(\mathcal{N}, \tau)$ , and if  $\Delta$  is a compact subset of  $(a, b)$ , then*

(i) *the function  $r \in [0, 1] \mapsto E_\Delta^{F_0+rK}$  takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)$  and is  $\mathcal{L}^1(\mathcal{N}, \tau)$ -bounded;*

(ii) *there exists  $R > 0$  such that the function  $K \in B_R \cap \mathcal{K}(\mathcal{N}, \tau) \mapsto E_\Delta^{F_0+K}$  takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)$  and is  $\mathcal{L}^1(\mathcal{N}, \tau)$ -bounded.*

*Proof.* (i) That  $E_\Delta^{F_r} = E_\Delta^{F_0+rK} \in \mathcal{L}^1(\mathcal{N}, \tau)$  follows from Lemmas 1.25 and 1.26. By (13) we have  $E_\Delta^{F_r} \leq c_0 E_{[a,b]}^{F_r} (b-F_r)(F_r-a)$  for all  $r \in [0, 1]$  and hence by [22, Lemma 2.5]

$$\mu_t(E_\Delta^{F_r}) \leq c_0 \mu_t \left( E_{[a,b]}^{F_r} (b-F_r)(F_r-a) \right) \leq c_0 \mu_t [(b-F_r)(F_r-a)].$$

Since  $(b-F_r)(F_r-a) = (b-F_0)(F_0-a) + rL_1 - r^2L_2$ , where  $L_1, L_2 \in \mathcal{K}(\mathcal{N}, \tau)$ , we have by [22, Lemma 2.5(v)]

$$(14) \quad \begin{aligned} \mu_t(E_\Delta^{F_r}) &\leq c_0 (\mu_{t/3}[(b-F_0)(F_0-a)] + r\mu_{t/3}(L_1) + r^2\mu_{t/3}(L_2)) \\ &\leq c_0 (\mu_{t/3}[(b-F_0)(F_0-a)] + \mu_{t/3}(L_1) + \mu_{t/3}(L_2)), \end{aligned}$$

so that  $\mu_t(E_\Delta^{F_r}) = \chi_{[0, \tau(E_\Delta^{F_r})]}(t)$  is majorized for all  $r \in [0, 1]$  by a single function decreasing to 0 when  $t \rightarrow \infty$ , since all three operators  $(b-F_0)(F_0-a)$ ,  $L_1$  and  $L_2$  are  $\tau$ -compact. The same argument as in Lemma 1.7 now completes the proof.

(ii) If  $F = F_0 + K$  then  $(b - F)(F - a) = (b - F_0)(F_0 - a) + L$ , where  $L = (b - F_0)K - K(F_0 - a) - K^2 \in \mathcal{K}(\mathcal{N}, \tau)$ . Choose the number  $R > 0$  such that  $\|K\| < R$  implies  $\|L\| < 1$ . Then by (14) the function  $t \mapsto \mu_t(E_{\Delta}^{F+K}) = \chi_{[0, \tau(E_{\Delta}^{F+K})]}(t)$  will be majorized by a single function decreasing to a number  $< 1$ , so that the same argument as in Lemma 1.7 again completes the proof.  $\square$

**Proposition 1.27.** *Let  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ ,  $K = K^* \in \mathcal{K}(\mathcal{N}, \tau)$ , and let  $h \in C_c^2(a, b)$ . Then*

- (i) *the function  $r \in \mathbb{R} \mapsto h(F_0 + rK)$  takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)$  and is  $\mathcal{L}^1(\mathcal{N}, \tau)$ -continuous;*
- (ii) *there exists  $R > 0$  such that the function  $K \in B_R \cap \mathcal{K}(\mathcal{N}, \tau) \mapsto h(F_0 + K)$  takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)$  and is  $\mathcal{L}^1(\mathcal{N}, \tau)$ -continuous;*

*Proof.* The proof of this proposition follows verbatim the proof of Proposition 1.20 with references to Lemmas 1.24, 1.25 and 1.26 instead of Lemmas 1.3, 1.5 and Corollary 1.8.  $\square$

**Lemma 1.28.** *If  $F_1, F_2 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ ,  $X \in \mathcal{K}(\mathcal{N}, \tau)_{sa}$  and  $h \in C_c^3(a, b)$ , then the double operator integral*

$$T_{h^{[1]}}^{F_1, F_2}(X)$$

*takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)$  and is  $\mathcal{L}^1(\mathcal{N}, \tau)$ -continuous with respect to norm perturbations of  $F_1$  and  $F_2$  by  $\tau$ -compact operators.*

The proof of this lemma is similar to that of Lemma 1.22 with references to Lemma 1.26(ii) and Proposition 1.27(ii) instead of Corollary 1.8 and Proposition 1.20.

**Theorem 1.29.** *Let  $\mathcal{N}$  be a semifinite von Neumann algebra. If  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ ,  $h \in C_c^3(a, b)$ , then the function  $h: F \in F_0 + \mathcal{K}_{sa}(\mathcal{N}, \tau) \mapsto h(F_0) + \mathcal{K}_{sa}(\mathcal{N}, \tau)$  takes values in  $\mathcal{L}^1(\mathcal{N}, \tau)_{sa}$ . Moreover, it is affinely  $(\mathcal{K}, \mathcal{L}^1)$ -Fréchet differentiable, the equality*

$$D_{\mathcal{K}, \mathcal{L}^1} h(F) = T_{h^{[1]}}^{F, F}$$

*holds, and  $D_{\mathcal{K}, \mathcal{L}^1} h(F)$  is  $(\mathcal{K}, \mathcal{L}^1)$  continuous, so that*

$$(15) \quad h(F_{r_1}) - h(F_{r_0}) = \int_{r_1}^{r_0} T_{h^{[1]}}^{F_r, F_r}(K) dr, \quad r_0, r_1 \in \mathbb{R},$$

*where  $K \in \mathcal{K}_{sa}(\mathcal{N}, \tau)$ ,  $F_r = F_0 + rK$  and the integral is in  $\mathcal{L}^1(\mathcal{N}, \tau)$ -norm.*

The proof is similar to that of Theorem 1.23 with use of Proposition 1.27(ii) and Lemma 1.28 instead of Proposition 1.20 and Lemma 1.22, and therefore it is omitted.

## 2. SPECTRAL SHIFT FUNCTION

We will take an approach to the notion of spectral shift function suggested by Birman-Solomyak formula (16). The key point is that once one appreciates that the spectral shift function of M.G.Krein is related to spectral flow in a specific fashion one can reformulate the whole approach to take advantage of what is known about spectral flow as expounded for example in [5]. The theorem in [3] which connects spectral flow and the spectral shift function contains the germ of the idea but one needs the technical machinery of the last Section to exploit this.

We now explain this different way to approach spectral shift theory which is influenced by ideas from noncommutative geometry.

### 2.1. The unbounded case.

**2.1.1. Spectral shift measure.** In order to make our main definition we need to prove a preliminary result which complements [3, Lemma 6.2]. The latter asserts that the function  $\gamma(\lambda, r) = \tau \left( V E_{\lambda}^{D_r} \right)$  is measurable for every  $V \in \mathcal{L}^1(\mathcal{N}, \tau)$  and  $D = D^* \eta \mathcal{N}$ .

**Lemma 2.1.** *Let  $(\mathcal{N}, \tau)$  be a semifinite von Neumann algebra and let  $D = D^* \eta \mathcal{N}$  have  $\tau$ -compact resolvent. If  $V = V^* \in \mathcal{N}$  then the function  $f: (a, b, r) \in \mathbb{R}^3 \mapsto \tau \left( V E_{(a,b)}^{D_r} \right)$  is measurable.*

*Proof.* W.l.o.g. we can assume that  $V \geq 0$ . It is enough to prove that the function  $f$  is measurable with respect to the second variable  $b$  and with respect to  $r$ . Since  $\tau \left( V E_{(a,b)}^{D_r} \right) = \tau \left( \sqrt{V} E_{(a,b)}^{D_r} \sqrt{V} \right)$ , we know by [27, Lemma 5.9] that it is enough to prove that the operator function  $(r, b) \mapsto \sqrt{V} E_{(a,b)}^{D_r} \sqrt{V}$  is  $so^*$ -measurable. By [3, Proposition 3.2] it is enough to prove that for any  $\xi, \eta \in \mathcal{H}$  the scalar function  $\left\langle \sqrt{V} E_{(a,b)}^{D_r} \sqrt{V} \xi, \eta \right\rangle = \text{Tr}(\theta_{\sqrt{V}\xi, \sqrt{V}\eta} E_{(a,b)}^{D_r})$  is measurable, where  $\theta_{\xi, \eta}(\zeta) := \langle \xi, \zeta \rangle \eta$ . Since the operator  $\theta_{\sqrt{V}\xi, \sqrt{V}\eta}$  is trace class, the measurability of this function follows from [3, Lemma 6.2].  $\square$

**Definition 2.2.** *If  $D_0 = D_0^* \eta \mathcal{N}$  has  $\tau$ -compact resolvent and if  $D_1 = D_0 + V$ ,  $V \in \mathcal{N}_{sa}$ , then the spectral shift measure for the pair  $(D_0, D_1)$  is defined to be the following Borel measure on  $\mathbb{R}$*

$$(16) \quad \Xi_{D_1, D_0}(\Delta) = \int_0^1 \tau \left( V E_{\Delta}^{D_r} \right) dr, \quad \Delta \in \mathcal{B}(\mathbb{R}).$$

*The generalized function*

$$(17) \quad \xi_{D_1, D_0}(\lambda) = \frac{d}{d\lambda} \Xi_{D_1, D_0}(a, \lambda)$$

is called the spectral shift distribution for the pair  $(D_0, D_1)$ .

Evidently, this definition does not depend on a choice of  $a$ . By Lemmas 1.3, 1.4, 2.1 and Corollary 1.9(i) the measure  $\Xi$  exists and is locally finite.

Our task now is to show that the spectral shift distribution is in fact a function of locally bounded variation.

**2.1.2. Spectral shift function.** The main result we wish to establish next is that the spectral shift measure is absolutely continuous with respect to Lebesgue measure. Moreover its density, which we previously referred to as the spectral shift distribution, is in fact a function of locally bounded variation which we will then refer to as the spectral shift function. It is our extension of M. G. Krein's function to the setting of this paper.

Our method of proof is to first establish some trace formulae.

**Lemma 2.3.** (i) Let  $D = D^* \eta \mathcal{N}$  have  $\tau$ -compact resolvent. A function  $\alpha \in B(\mathbb{R})$  is 1-summable with respect to the measure  $\tau(E_\Delta^D)$  ( $\Delta \in \mathcal{B}(\mathbb{R})$ ), if and only if  $\alpha(D) \in \mathcal{L}^1(\mathcal{N}, \tau)$  and in this case

$$\tau(\alpha(D)) = \int_{\mathbb{R}} \alpha(\lambda) \tau(dE_\lambda^D).$$

Furthermore, for any  $V = V^* \in \mathcal{N}$  the function  $\alpha$  is 1-summable with respect to the measure  $\tau(VE_\Delta^D)$ , and

$$\tau(V\alpha(D)) = \int_{\mathbb{R}} \alpha(\lambda) \tau(VdE_\lambda^D).$$

(ii) Let  $F \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ . A function  $\alpha \in B(a, b)$  is 1-summable with respect to the measure  $\tau(E_\Delta^F)$  ( $\Delta \in \mathcal{B}(a, b)$ ), if and only if  $\alpha(F) \in \mathcal{L}^1(\mathcal{N}, \tau)$  and in this case

$$\tau(\alpha(F)) = \int_a^b \alpha(\lambda) \tau(dE_\lambda^F).$$

Furthermore, for any  $V = V^* \in \mathcal{N}$  the function  $\alpha$  is 1-summable with respect to the measure  $\tau(VE_\Delta^F)$ , and

$$\tau(V\alpha(F)) = \int_a^b \alpha(\lambda) \tau(VdE_\lambda^F).$$

*Proof.* We give only the proof of (i). W.l.o.g. we can assume that  $\alpha$  is a non-negative function. If  $\alpha$  is a simple function then the first part of the claim follows from Lemma 1.4. Let  $\alpha_n$  be an increasing sequence of simple non-negative functions, converging pointwise to  $\alpha$ .

Then for each of the functions  $\alpha_n$  the first equality is true. The supremum of the increasing sequence of non-negative operators  $\alpha_n(D)$  is  $\alpha(D)$  and the supremum of the increasing sequence of numbers  $\int_{\mathbb{R}} \alpha_n(\lambda) \tau(dE_{\lambda}^D)$  is  $\int_{\mathbb{R}} \alpha(\lambda) \tau(dE_{\lambda}^D)$ . Hence, both non-negative numbers  $\int_{\mathbb{R}} \alpha(\lambda) \tau(dE_{\lambda}^D)$  and  $\tau(\alpha(D))$  are finite or infinite simultaneously, which proves the first part of the lemma.

For the second part we can assume w.l.o.g. that  $V \geq 0$ . Then again the both parts of the second equality make sense and they are equal for simple functions.

Since the measure  $\tau(VE_{\Delta}^D) = \tau(\sqrt{V}E_{\Delta}^D\sqrt{V})$ ,  $\Delta \in \mathcal{B}(\mathbb{R})$ , is non-negative and the supremum of  $\sqrt{V}\alpha_n(D)\sqrt{V} \in \mathcal{L}^1(\mathcal{N}, \tau)$  is  $\sqrt{V}\alpha(D)\sqrt{V}$  we have that

$$\begin{aligned} \int_{\mathbb{R}} \alpha(\lambda) \tau(VdE_{\lambda}^D) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \alpha_n(\lambda) \tau(VdE_{\lambda}^D) \\ &= \lim_{n \rightarrow \infty} \tau(\sqrt{V}\alpha_n(D)\sqrt{V}) = \tau(\sqrt{V}\alpha(D)\sqrt{V}), \end{aligned}$$

so that  $\int_{\mathbb{R}} \alpha(\lambda) \tau(VdE_{\lambda}^D)$  and  $\tau(V\alpha(D)) = \tau(\sqrt{V}\alpha(D)\sqrt{V})$  are finite or infinite simultaneously.  $\square$

We need the following version of Fubini's theorem.

**Lemma 2.4.** (i) For any self-adjoint operator  $D\eta\mathcal{N}$  with  $\tau$ -compact resolvent and  $V = V^* \in \mathcal{N}$ , let  $m_{D,V}(\Delta) = \tau(VE_{\Delta}^D)$ . Let  $D_0 = D_0^*\eta\mathcal{N}$  have  $\tau$ -compact resolvent and let  $D_r = D_0 + rV$ . If  $g \in B_c(\mathbb{R})$ , then

$$(18) \quad \int_0^1 dr \int_{\mathbb{R}} g(\lambda) m_{D_r,V}(d\lambda) = \int_{\mathbb{R}} g(\lambda) \Xi_{D_1,D_0}(d\lambda).$$

(ii) For any  $F \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$  and  $V = V^* \in \mathcal{N}$ , let  $m_{F,V}(\Delta) = \tau(VE_{\Delta}^F)$ ,  $\Delta \in \mathcal{B}(a,b)$ . Let  $F \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$  and let  $F_r = F_0 + rV$ . If  $g \in B_c(a,b)$ , then

$$\int_0^1 dr \int_a^b g(\lambda) m_{F_r,V}(d\lambda) = \int_a^b g(\lambda) \Xi_{F_1,F_0}(d\lambda).$$

*Proof.* (See also [25, VI.2]). We give only the proof of (i). The measurability of the function  $r \mapsto \int_{\mathbb{R}} g(\lambda) m_{D_r,V}(d\lambda)$  follows from Lemma 2.1.

Note, that both integrals are repeated ones. Let  $\Omega \supseteq \text{supp}(g)$  be a finite interval. By Corollary 1.9 (i) there exists  $M > 0$  such that for all  $r \in [0, 1]$  we have  $|m_{D_r,V}(\Omega)| \leq M$ .

If  $g(\lambda) = \chi_\Delta(\lambda)$ ,  $\Delta \in \mathcal{B}(\Omega)$ , then

$$\int_0^1 dr \int_\Omega \chi_\Delta(\lambda) m_{D_r, V}(d\lambda) = \int_0^1 m_{D_r, V}(\Delta) dr = \Xi(\Delta) = \int_\Omega \chi_\Delta(\lambda) \Xi(d\lambda).$$

So, (18) is true for simple functions. Let now  $g$  be an arbitrary function from  $B_c(\Omega)$ , let  $\varepsilon > 0$  and let  $h$  be a simple function such that  $\|g - h\|_\infty < \varepsilon$ . Then the LHS of (18) is equal to

$$\int_0^1 dr \int_\Omega (g - h)(\lambda) m_{D_r, V}(d\lambda) + \int_0^1 dr \int_\Omega h(\lambda) m_{D_r, V}(d\lambda) = (I) + (II),$$

and the RHS of (18) is equal to

$$\int_\Omega (g - h)(\lambda) \Xi(d\lambda) + \int_\Omega h(\lambda) \Xi(d\lambda) = (III) + (IV).$$

We have  $(II) = (IV)$ . Further,  $|(I)| \leq M \|g - h\|_\infty \leq M\varepsilon$  and  $|(III)| \leq M \|g - h\|_\infty \leq M\varepsilon$ .  $\square$

The following theorem complements [3, Theorem 6.3].

**Theorem 2.5.** *If  $D = D^* \eta \mathcal{N}$  has  $\tau$ -compact resolvent, if  $V = V^* \in \mathcal{N}$ , and if  $D_1 = D_0 + V$ , then the measure  $\Xi_{D_1, D_0}$  is absolutely continuous, its density is equal to*

$$(19) \quad \xi_{D_1, D_0}(\cdot) = \tau \left( E_{(a, \lambda]}^{D_0} \right) - \tau \left( E_{(a, \lambda]}^{D_1} \right) + \text{const}$$

for almost all  $\lambda \in \mathbb{R}$ . Moreover, for all  $f \in C_c^3(\mathbb{R})$   $f(D_1) - f(D_0) \in \mathcal{L}^1(\mathcal{N}, \tau)$  and

$$(20) \quad \tau(f(D_1) - f(D_0)) = \int_{\mathbb{R}} f'(\lambda) \xi_{D_1, D_0}(\lambda) d\lambda.$$

*Proof.* By Lemma 1.3 and Corollary 1.5  $f(D_1) - f(D_0) \in \mathcal{L}^1(\mathcal{N}, \tau)$ .

By Lemma 1.2 we need only consider the case of a non-negative function  $f$  with  $g := \sqrt{f} \in C_c^2(\mathbb{R})$ .

We have by (12)

$$f(D_1) - f(D_0) = \int_0^1 T_{f[1]}^{D_r, D_r}(V) dr,$$

where the integral converges in  $\mathcal{L}^1(\mathcal{N}, \tau)$ -norm. Hence, it follows from Lemma 1.17 that

$$(21) \quad \begin{aligned} f(D_1) - f(D_0) &= \int_0^1 \int_\Pi (\alpha_1(D_r, \sigma) V \beta_1(D_r, \sigma) + \alpha_2(D_r, \sigma) V \beta_2(D_r, \sigma)) d\nu_g(\sigma) dr. \end{aligned}$$

Now, for a fixed  $\sigma \in \Pi$ , we have

$$\begin{aligned} & \tau(\alpha_1(D_r, \sigma)V\beta_1(D_r, \sigma) + \alpha_2(D_r, \sigma)V\beta_2(D_r, \sigma)) \\ &= \tau(V(\alpha_1(D_r, \sigma)\beta_1(D_r, \sigma) + \alpha_2(D_r, \sigma)\beta_2(D_r, \sigma))) \\ &= \int_{\mathbb{R}} (\alpha_1(\lambda, \sigma)\beta_1(\lambda, \sigma) + \alpha_2(\lambda, \sigma)\beta_2(\lambda, \sigma)) \tau(VdE_{\lambda}^{D_r}), \end{aligned}$$

where the last equality uses Lemma 2.3. (That  $\alpha_1(\lambda)\beta_1(\lambda) + \alpha_2(\lambda)\beta_2(\lambda)$  belongs to  $B_c(\mathbb{R})$  follows from Lemma 1.14)

Hence using (21), and by Lemma 1.11 applied to the finite measure space  $([0, 1] \times \Pi, dr \times \nu_g)$  our previous equality implies that we have:

$$\begin{aligned} A &:= \tau(f(D_1) - f(D_0)) \\ &= \int_0^1 \int_{\Pi} \tau(\alpha_1(D_r, \sigma)V\beta_1(D_r, \sigma) + \alpha_2(D_r, \sigma)V\beta_2(D_r, \sigma)) d\nu_g(\sigma) dr \\ &= \int_0^1 \int_{\Pi} \int_{\mathbb{R}} (\alpha_1(\lambda, \sigma)\beta_1(\lambda, \sigma) + \alpha_2(\lambda, \sigma)\beta_2(\lambda, \sigma)) \tau(VdE_{\lambda}^{D_r}) d\nu_g(\sigma) dr. \end{aligned}$$

Now, by Lemma 2.3, Fubini's theorem, and Lemma 1.14 we have

$$\begin{aligned} A &= \int_0^1 \int_{\mathbb{R}} \int_{\Pi} (\alpha_1(\lambda, \sigma)\beta_1(\lambda, \sigma) + \alpha_2(\lambda, \sigma)\beta_2(\lambda, \sigma)) d\nu_g(\sigma) \tau(VdE_{\lambda}^{D_r}) dr \\ &= \int_0^1 \int_{\mathbb{R}} f'(\lambda) \tau(VdE_{\lambda}^{D_r}) dr. \end{aligned}$$

Finally, by Lemma 2.4 we have

$$(22) \quad A = \int_{\mathbb{R}} f'(\lambda) \int_0^1 \tau(VdE_{\lambda}^{D_r}) dr = \int_{\mathbb{R}} f'(\lambda) d\Xi_{D_1, D_0}(\lambda).$$

Let  $f \in C_c^1(\mathbb{R})$  and take a point  $a$  outside of the support of  $f$ . Then we have (see [1, Proposition 8.5.5])

$$\begin{aligned} (23) \quad A &= \tau(f(D_1) - f(D_0)) = \tau(f(D_1)) - \tau(f(D_0)) \\ &= \int_{\mathbb{R}} f(\lambda) d\tau(E_{(a, \lambda]}^{D_1}) - \int_{\mathbb{R}} f(\lambda) d\tau(E_{(a, \lambda]}^{D_0}) \quad (\text{integrating by parts}) \\ &= - \int_{\mathbb{R}} f'(\lambda) \left( \tau(E_{(a, \lambda]}^{D_1}) - \tau(E_{(a, \lambda]}^{D_0}) \right) d\lambda. \end{aligned}$$

Comparing (23) and (22) we see that  $\Xi$  is absolutely continuous with density equal to

$$(24) \quad \xi_{D_1, D_0}(\lambda) = \tau(E_{(a, \lambda]}^{D_0}) - \tau(E_{(a, \lambda]}^{D_1}) + \text{const}.$$

□

It is worth noting that the formula (20) does not determine the function  $\xi$  uniquely, but only up to an additive constant.

**Remark 1.** *This theorem is an analogue of [4, Theorem 3.1], in which the existence and absolute continuity of the spectral shift measure were proved for any self-adjoint operator  $D$  affiliated with  $\mathcal{N}$  and  $\tau$ -trace class operator  $V \in \mathcal{N}$ .*

As a result of what we have proved to this point we are now in a position to assert that in fact the spectral shift distribution is an everywhere defined function and hence to change our terminology and refer to  $\xi$  as a function. Moreover this last lemma enables one to modify  $\xi$  so as to make it a function defined everywhere in a natural way.

**Definition 2.6.** *If the expression (24) is continuous at a point  $\lambda \in \mathbb{R}$ , then we define  $\xi_{D_1, D_0}(\lambda)$  via formula (24). Otherwise, we define the value of the spectral shift function  $\xi$  at a discontinuity point to be half sum of left and right limits.*

**Corollary 2.7.** *The spectral shift function  $\xi$  is a function of locally bounded variation.*

*Proof.* This is immediate because  $\xi$  is the difference of two increasing functions by the last formula.  $\square$

**Lemma 2.8.** *Let  $D_0 \eta \mathcal{N}$  be a self-adjoint operator with  $\tau$ -compact resolvent, let  $V \in \mathcal{N}_{sa}$  and let  $D_1 = D_0 + V$ . If  $f \in B_c(\mathbb{R})$  then*

$$\int_{-\infty}^{\infty} f(\lambda) \xi_{D_1, D_0}(\lambda) d\lambda = \int_0^1 \tau(Vf(D_r)) dr.$$

*Proof.* It follows from Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned} \int_0^1 \tau(Vf(D_r)) dr &= \int_0^1 \int_{-\infty}^{\infty} f(\lambda) \tau(VdE_{\lambda}^{D_r}) d\lambda dr \\ &= \int_{-\infty}^{\infty} f(\lambda) \int_0^1 \tau(VdE_{\lambda}^{D_r}) dr d\lambda. \end{aligned}$$

$\square$

**2.1.3. The spectral shift function for unitarily equivalent operators.** The situation where the operators  $D$  and  $D + V$ , are unitarily equivalent arises naturally in noncommutative geometry in the context of spectral triples. One thinks of the unitary implementing the equivalence as a gauge transformation by analogy with the study of gauge transformations of Dirac type operators. It thus warrants special consideration especially in view of our first result below.

**Theorem 2.9.** *Let  $D$  be a self-adjoint operator affiliated with  $\mathcal{N}$  having  $\tau$ -compact resolvent and let  $V = V^* \in \mathcal{N}$  be such that the operators  $D + V$*



and  $D$  are unitarily equivalent. Then the spectral shift function  $\xi_{D+V,D}$  of the pair  $(D+V, D)$  is constant on  $\mathbb{R}$ .

*Proof.* The operators  $f(D+V)$  and  $f(D)$  are unitarily equivalent and for  $f \in C_c^\infty(\mathbb{R})$  they are  $\tau$ -trace class by Corollary 1.5. Hence,

$$\tau(f(D+V) - f(D)) = 0,$$

so that by Theorem 2.5 the equality

$$\int_{\mathbb{R}} f'(\lambda) \xi_{D+V,D}(\lambda) d\lambda = 0$$

holds for any  $f \in C_c^\infty(\mathbb{R})$ . Now, integration by parts shows that  $\xi'(\lambda)$  is zero as generalized function on  $\mathbb{R}$ , which by [23, Ch. I.2.6] implies that  $\xi$  is equal to a constant function.  $\square$

Note, function  $\xi$  in this theorem is equal to a constant function everywhere, not just almost everywhere.

Our second major result on the spectral shift function in this special context is the following theorem. We shall show in Section 3 below that this theorem extends one of the main results of [16].

**Theorem 2.10.** *Let  $D_0$  be a self-adjoint operator with  $\tau$ -compact resolvent, affiliated with  $\mathcal{N}$ . Let  $V = V^* \in \mathcal{N}$  be such that the operators  $D_1 = D_0 + V$  and  $D_0$  are unitarily equivalent. If  $f \in C_c^2(\mathbb{R})$  then*

$$(25) \quad \xi_{D_1,D_0}(\mu) = C^{-1} \int_0^1 \tau(Vf(D_r - \mu)) dr, \quad \forall \mu \in \mathbb{R},$$

where  $C = \int_{\mathbb{R}} f(\lambda) d\lambda$ .

*Proof.* For any fixed  $\mu$  the operator  $D_r - \mu$  has  $\tau$ -compact resolvent by Lemma 1.3 and the function  $r \mapsto \tau(Vf(D_r - \mu))$  is continuous by Proposition 1.20, so that the integral on the RHS of (25) exists. By Lemma 2.8 and Theorem 2.9 we have

$$\int_0^1 \tau(Vf(D_r - \mu)) dr = \int_{\mathbb{R}} f(\lambda - \mu) \xi_{D_1,D_0}(\lambda) d\lambda = \xi(0) \int_{\mathbb{R}} f(\lambda) d\lambda.$$

$\square$

**2.2. The bounded case.** As we remarked previously, our technique in the next Section for discussing spectral flow in the unbounded case is to map into the space of bounded  $\tau$ -Fredholm operators. We thus need to develop the theory described in the previous subsections *ab initio* for the bounded case. Fortunately this is not a difficult task as the proofs are much the same. As we will see, because we are considering bounded perturbations of our unbounded operators, it suffices to consider compact perturbations in the bounded case.

**Definition 2.11.** If  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ ,  $K \in \mathcal{K}(\mathcal{N}, \tau)_{sa}$ ,  $F_1 = F_0 + K$  and if  $F_r = F_0 + rK$ , then the spectral shift measure for the pair  $(F_0, F_1)$  is defined to be the following Borel measure on  $(a, b)$

$$(26) \quad \Xi_{F_1, F_0}(\Delta) = \int_0^1 \tau \left( K E_{\Delta}^{F_r} \right) dr, \quad \Delta \in \mathcal{B}(a, b).$$

The generalized function

$$(27) \quad \xi_{F_1, F_0}(\lambda) = \frac{d}{d\lambda} \Xi_{F_1, F_0}(c, \lambda), \quad c \in (a, b),$$

is called the spectral shift distribution for the pair  $(F_0, F_1)$ .

Evidently, this definition does not depend on a choice of  $c \in (a, b)$ . The measurability of the function  $r \mapsto \tau \left( K E_{\Delta}^{F_r} \right)$  may be established following the argument of Lemma 2.1, using Lemma 1.26. It follows that the measure  $\Xi$  exists and is locally-finite on  $(a, b)$ .

**Proposition 2.12.** If  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ ,  $K \in \mathcal{K}(\mathcal{N}, \tau)$  and if  $F_1 = F_0 + K$ , then

(i) the measure  $\Xi_{F_1, F_0}$  is absolutely continuous and its density is equal to

$$\xi_{F_1, F_0}(\lambda) = \tau \left( E_{(c, \lambda]}^{F_0} - E_{(c, \lambda]}^{F_1} \right) + \text{const}, \quad \lambda \in (c, b),$$

where  $c$  is an arbitrary number from  $(a, b)$ ;

(ii) there exists a unique function  $\xi_{F_1, F_0}(\cdot)$  of locally bounded variation on  $(a, b)$ , such that for any  $h \in C_c^2(a, b)$  the following equality holds true

$$\tau(h(F_1) - h(F_0)) = \int_a^b h'(\lambda) \xi_{F_1, F_0}(\lambda) d\lambda.$$

The proof is identical to the proof of Theorem 2.5, with references to 1.24, 1.25, (15) instead of 1.3, 1.5, (12) and hence we omit it.

**Corollary 2.13.** In the setting of Proposition 2.12, if  $F_0$  and  $F_1$  are unitarily equivalent, then  $\xi_{F_1, F_0}$  is constant on  $(a, b)$ .

The proof is similar to the proof of Theorem 2.9.

**Definition 2.14.** We redefine the function  $\xi_{F_1, F_0}$  at discontinuity points to be half the sum of the left and the right limits of the RHS of the last equality.

Thus, the function  $\xi_{F_1, F_0}$  is defined everywhere on  $(a, b)$ .

**Lemma 2.15.** If  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ ,  $K \in \mathcal{K}(\mathcal{N}, \tau)$ , if  $F_r = F_0 + rK$ ,  $r \in [0, 1]$  and if  $h \in B_c(a, b)$  then

$$\int_{\mathbb{R}} h(\lambda) \xi_{F_1, F_0}(\lambda) d\lambda = \int_0^1 \tau(Kh(F_r)) dr.$$

This Lemma and its proof are bounded variants of Lemma 2.8, so we omit the details.

### 3. SPECTRAL FLOW

**3.1. The spectral flow function.** We wish to avoid a long excursion into the analytic theory of spectral flow preferring the reader to read the early Sections of [5] for the relevant background and definitions. With those prerequisites it is possible to appreciate that the following notions are well defined.

We now introduce the spectral flow function on the real line. As usual we have  $D_0 = D_0^* \eta \mathcal{N}$  with  $\tau$ -compact resolvent and  $D_1 = D_0 + V$  with  $V$  bounded. Then we define the spectral flow function to be

$$\mu \mapsto \text{sf}(\mu, D_0, D_1), \quad \mu \in \mathbb{R},$$

where  $\text{sf}(\mu, D_0, D_1)$  is spectral flow from  $D_0 - \mu$  to  $D_1 - \mu$  along the path  $D_r - \mu = D_0 - \mu + rV$ . We remark that the homotopy invariance of spectral flow means that, in the affine space of bounded perturbations of a fixed operator  $D_0$ , spectral flow does not depend on the choice of continuous path but only on the endpoints. However one does need to make precise what one means by continuity in a path parameter in the unbounded case. We define the space of self adjoint, unbounded  $\tau$ -Fredholm operators to be those operators that under the map

$$D \mapsto F_D = D(1 + D)^{-1/2}$$

are sent to bounded  $\tau$ -Fredholm operators in  $\mathcal{N}$ . This definition is equivalent to the usual definition for unbounded Fredholm operators in a semifinite von Neumann algebra (more details on this can be found in [18].)

A path of unbounded  $\tau$ -Fredholm operators is said to be continuous if its image under this map is continuous in the norm topology on the bounded  $\tau$ -Fredholm operators. This topology is usually called the Riesz topology and it is different from the graph norm topology used in [12]. A more detailed discussion of topologies on the set of unbounded self-adjoint Fredholm operators and the relevance of these for spectral flow may be found in [36].

We note that the condition of having compact resolvent implies the  $\tau$ -Fredholm property for the unbounded operator.

We now recall, to provide some motivation for the point of view of this Section, some ideas from [16] which is couched in the language of noncommutative geometry [19]. In [16] the condition of theta summability ( $\tau(e^{-tD_0^2}) < \infty$  for all real  $t > 0$ ) is imposed and then the main result of [16] is an analytic formula for spectral flow from  $D_0$  to  $D_1 = D_0 + V$ . The ideas behind this

formula go back to [2] and a more complete history may be found in [5]. The formula of [16] is as follows

$$(28) \quad \text{sf}(D_0, D_1) = \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau \left( V e^{-\varepsilon D_r^2} \right) dr \\ + \frac{1}{2} (\eta_\varepsilon(D_1) - \eta_\varepsilon(D_0)) + \frac{1}{2} \tau ([\ker(D_1)] - [\ker(D_0)]),$$

where  $\eta_\varepsilon$  is a ‘truncated eta invariant’. For an unbounded self-adjoint operator  $D$  for which  $e^{-tD^2}$  is  $\tau$ -trace class for all  $t > 0$ , it is defined in [16, Definition 8.1] following [24] by

$$\eta_\varepsilon(D) := \frac{1}{\sqrt{\pi}} \int_\varepsilon^\infty \tau \left( D e^{-tD^2} \right) t^{-1/2} dt.$$

When the endpoints are unitarily equivalent the two  $\eta_\varepsilon$  terms and the kernel correction terms in the formula for spectral flow cancel.

In [3] we showed that in the theta summable case and for trace class perturbations the spectral shift function and the spectral flow function differ only by kernel correction terms. Our aim in this paper is to show that this is the case more generally and in fact to go further. We will demonstrate that the spectral shift function provides a way to prove more general analytic formulae for spectral flow than is achieved in [16].

The plan of this Section is to first establish a geometric framework that is analogous to that of Getzler [24] and [16]. Then we derive analytic formulae for spectral flow in terms of the spectral shift function in both the case of paths of bounded  $\tau$ -Fredholm operators and for paths of unbounded  $\tau$ -Fredholm operators. The starting point is basically the same as that of [16] in that we need a formula for the relative index of two projections. This next result is a strengthening of [16, Theorem 4.1].

**Lemma 3.1.** *Let  $P$  and  $Q$  be two projections in the semifinite von Neumann algebra  $\mathcal{N}$  and let  $a < 0 < b$  be two real numbers. Let  $\kappa$  be a continuous function such that for any  $s \in [0, \frac{(b-a)^2}{4}]$   $\kappa(s(P-Q)^2)$  is  $\tau$ -trace class. Then  $F_0 = (b-a)P + a$  and  $F_1 = (b-a)Q + a$  are self-adjoint  $\tau$ -Fredholm operators from  $\mathcal{F}^{a,b}(\mathcal{N}, \tau)$  as is the path  $F_r = F_0 + r(F_1 - F_0)$ , and*

$$\text{sf}(\{F_r\}) = C_{a,b}^{-1} \int_0^1 \tau \left( \dot{F}_r \kappa[(b - F_r)(F_r - a)] \right) dr,$$

where  $C_{a,b} = \int_0^1 (b-a) \kappa((b-a)^2(r-r^2)) dr$  is a constant, and the derivative  $\dot{F}_r$  is  $\|\cdot\|$ -derivative.

*Proof.* We have

$$\dot{F}_r = F_1 - F_0 = (b-a)(Q - P)$$

and

$$(b - F_r)(F_r - a) = (b-a)^2 r(1-r)(Q - P)^2,$$

so that by assumption  $\kappa[(b - F_r)(F_r - a)]$  is  $\tau$ -trace class for  $r \in [0, 1]$ . For each  $r \in (0, 1)$  define

$$f_r(x) = (b - a)x\kappa((b - a)^2(r - r^2)x^2).$$

Then

$$\begin{aligned} (29) \quad & \int_0^1 \tau \left( \dot{F}_r \kappa[(b - F_r)(F_r - a)] \right) dr \\ &= \int_0^1 \tau \left( (b - a)(Q - P) \kappa[(b - a)^2 r(1 - r)(Q - P)^2] \right) dr, \end{aligned}$$

and by [16, Theorem 3.1] we have

$$\begin{aligned} & \int_0^1 \tau \left( \dot{F}_r \kappa[(b - F_r)(F_r - a)] \right) dr = \int_0^1 \tau(f_r(Q - P)) dr \\ &= \int_0^1 f_r(1) \operatorname{ind}(QP) dr = C_{a,b} \operatorname{ind}(QP) = C_{a,b} \operatorname{sf}(\{F_r\}), \end{aligned}$$

where the last equality follows from  $a < 0 < b$  and the definition of spectral flow.  $\square$

**Remark 2.** *In the proof of this lemma we use [16, Theorem 3.1] for functions  $f$  without the condition  $f(1) \neq 0$ . But an inspection of the proof of this theorem shows that this condition becomes superfluous, if we rewrite the statement of this theorem as  $f(1) \operatorname{ind}(QP) = \tau[f(P - Q)]$ , which for functions  $f$  with  $f(1) = 0$  becomes just  $0 = 0$ .*

**3.2. Spectral flow one-forms in the unbounded case.** The strategy of [16] is geometric and follows ideas of [24]. The first step in this strategy is summarized in Proposition 3.3 in preparation for which we need an explicit formula for the derivative of function of a path of operators. The method by which this is achieved in [16] does not apparently generalise sufficiently far to cover the situations considered in this paper. The double operator integral approach of Section 2 overcomes this problem.

**Lemma 3.2.** *Let  $D = D^* \eta \mathcal{N}$ , let  $X, Y \in \mathcal{N}$  and let  $f \in C_c^2(\mathbb{R})$  be a non-negative function such that  $g := \sqrt{f} \in C_c^2(\mathbb{R})$ . If  $YT_{f[1]}^{D,D}(X)$  and  $XT_{f[1]}^{D,D}(Y)$  are both  $\tau$ -trace class then*

$$\tau \left( YT_{f[1]}^{D,D}(X) \right) = \tau \left( XT_{f[1]}^{D,D}(Y) \right).$$

*Proof.* By Lemma 1.17 we have

$$\begin{aligned}
A &= \tau \left( Y T_{f^{[1]}}^{D,D}(X) \right) \\
&= \tau \left( Y \int_{\Pi} (\alpha_1(D, \sigma) X \beta_1(D, \sigma) + \alpha_2(D, \sigma) X \beta_2(D, \sigma)) d\nu_g(\sigma) \right) \\
&= \tau \left( Y \int_{\Pi} \left( e^{i(s_1-s_0)D} g(D) X e^{is_1 D} + e^{i(s_1-s_0)D} X g(D) e^{is_1 D} \right) d\nu_g(s_0, s_1) \right).
\end{aligned}$$

Making the change of variables  $s_1 - s_0 = t_0$ ,  $s_1 = t_1$ , and using (4) we have

$$\begin{aligned}
&\int_{\Pi} (\alpha_1(D, \sigma) X \beta_1(D, \sigma) + \alpha_2(D, \sigma) X \beta_2(D, \sigma)) d\nu_g(\sigma) \\
&= \operatorname{sgn}(t_0 + t_1) \frac{i}{\sqrt{2\pi}} \int_{\{(t_0, t_1) \in \mathbb{R}^2, t_0 t_1 \geq 0\}} \left[ e^{it_0 D} g(D) X e^{it_1 D} \right. \\
&\quad \left. + e^{it_0 D} X g(D) e^{it_1 D} \right] \mathcal{F}(g)(t_0 + t_1) dt_0 dt_1 \\
&= \operatorname{sgn}(t) \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \left( \int_{\Sigma_t} \left( e^{it_0 D} g(D) X e^{it_1 D} \right. \right. \\
&\quad \left. \left. + e^{it_0 D} X g(D) e^{it_1 D} \right) dl_t \right) \mathcal{F}(g)(t) dt,
\end{aligned}$$

where  $\Sigma_t = \{(t_0, t_1) \in \mathbb{R}^2 : t_0 t_1 \geq 0, t_0 + t_1 = t\}$  and  $dl_t$  is the Lebesgue measure on  $\Sigma_t$ . Thus, by Fubini's theorem [3, Lemma 3.8]

$$\begin{aligned}
A &= \operatorname{sgn}(t) \frac{i}{\sqrt{2\pi}} \tau \left( Y \int_{\mathbb{R}} \left( \int_{\Sigma_t} \left( e^{it_0 D} g(D) X e^{it_1 D} \right. \right. \right. \\
&\quad \left. \left. + e^{it_0 D} X g(D) e^{it_1 D} \right) dl_t \right) \mathcal{F}(g)(t) dt \Big) \\
&= \operatorname{sgn}(t) \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \left( \int_{\Sigma_t} \tau \left( Y e^{it_0 D} g(D) X e^{it_1 D} \right. \right. \\
&\quad \left. \left. + Y e^{it_0 D} X g(D) e^{it_1 D} \right) dl_t \right) \mathcal{F}(g)(t) dt \\
&= \operatorname{sgn}(t) \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \left( \int_{\Sigma_t} \tau \left( X e^{it_1 D} Y g(D) e^{it_0 D} \right. \right. \\
&\quad \left. \left. + X e^{it_1 D} g(D) Y e^{it_0 D} \right) dl_t \right) \mathcal{F}(g)(t) dt,
\end{aligned}$$

where the trace and the integral can be interchanged by Lemma 1.11. The integral above coincides with  $\tau \left( X T_{f^{[1]}}^{D,D}(Y) \right)$ .  $\square$

The key geometric idea is to regard the analytic formula for spectral flow of [24] and [16] as expressing it as an integral of a one form. As we are dealing with an affine space the geometry is easy to invoke as we see in the next result.

**Proposition 3.3.** *Let  $D$  be a self-adjoint operator affiliated with  $\mathcal{N}$ , having  $\tau$ -compact resolvent and let  $f \in C_c^3(\mathbb{R})$ . Let  $\alpha = \alpha^f$  be a 1-form on the affine space  $D_0 + \mathcal{N}_{sa}$  defined at the point  $D \in D_0 + \mathcal{N}_{sa}$  by the formula*

$$(30) \quad \alpha_D^f(X) = \tau(Xf(D)), \quad X \in \mathcal{N}_{sa}, \quad D \in D_0 + \mathcal{N}_{sa}.$$

*Then  $\alpha$  is a closed 1-form, and, hence, also exact by the Poincaré lemma.*

*Proof.* The proof follows mainly the lines of [15], with necessary adjustments. As usual, we can assume that  $f \geq 0$  and  $g := \sqrt{f} \in C_c^3(\mathbb{R})$ . We note that the operator  $Xf(D)$  is  $\tau$ -trace class, so that the 1-form  $\alpha$  is well-defined. Now, by the definition of the exterior differential, for  $X, Y \in \mathcal{N}$ , we have

$$d\alpha_D(X, Y) = \mathcal{L}_X \alpha_D(Y) - \mathcal{L}_Y \alpha_D(X) - \alpha_D([X, Y]),$$

where  $\mathcal{L}_X$  is the Lie derivative along the constant vector field  $X$ . Since the space  $D_0 + \mathcal{N}_{sa}$  is flat, we have  $[X, Y] = 0$ . So, we have to prove that  $\mathcal{L}_X \alpha_D(Y) = \mathcal{L}_Y \alpha_D(X)$ . It follows from Theorem 1.23 that

$$\begin{aligned} A := \mathcal{L}_X \alpha_D(Y) &= \left. \frac{d}{ds} \right|_{s=0} \alpha_{D+sX}(Y) = \left. \frac{d}{ds} \right|_{s=0} \tau(Yf(D+sX)) \\ &= \tau(YD_{\mathcal{N}, \mathcal{L}^1} f(D)(X)) = \tau\left(Y T_{f[1]}^{D,D}(X)\right). \end{aligned}$$

Hence, by Lemma 3.2

$$\mathcal{L}_X \alpha_D(Y) = \tau\left(Y T_{f[1]}^{D,D}(X)\right) = \tau\left(X T_{f[1]}^{D,D}(Y)\right) = \mathcal{L}_Y \alpha_D(X),$$

which implies that  $\alpha_D$  is a closed 1-form.  $\square$

Though closedness of a 1-form already should imply its exactness by the Poincaré lemma and contractibility of the domain we follow [16] and give an independent proof of exactness.

**Definition 3.4.** *Let  $D_0$  be a fixed self-adjoint operator with  $\tau$ -compact resolvent affiliated with  $\mathcal{N}$ , and let  $f \in C_c(\mathbb{R})$ . We define the function  $\theta^f$  on the affine space  $D_0 + \mathcal{N}$  by the formula*

$$\theta_D^f = \int_0^1 \tau(Vf(D_r)) \, dr,$$

*where  $D \in D_0 + \mathcal{N}_{sa}$ ,  $V = D - D_0$  and  $D_r = D_0 + rV$ . Measurability of the function  $r \mapsto \tau(Vf(D_r))$  follows from Lemma 2.1.*

**Proposition 3.5.** *Let  $f \in C_c^3(\mathbb{R})$  and let  $X \in \mathcal{N}$ . Then*

$$d\theta_D^f(X) = \alpha_D^f(X).$$

*Proof.* W.l.o.g. we can assume that  $X$  is self-adjoint. By definitions

$$\begin{aligned}
(A) &:= d\theta_D^f(X) \\
&= \frac{d}{ds} \Big|_{s=0} \theta_{D+sX}^f = \frac{d}{ds} \Big|_{s=0} \int_0^1 \tau((V+sX)f(D_r+srX)) dr \\
&= \lim_{s \rightarrow 0} \frac{1}{s} \int_0^1 \tau((V+sX)f(D_r+srX) - Vf(D_r)) dr \\
&= \lim_{s \rightarrow 0} \int_0^1 \tau(Xf(D_r+srX)) dr + \lim_{s \rightarrow 0} \frac{1}{s} \int_0^1 \tau(V(f(D_r+srX) - f(D_r))) dr.
\end{aligned}$$

The first summand of this sum by Proposition 1.20 is equal to

$$\int_0^1 \tau(Xf(D_r)) dr.$$

By Proposition 1.18(i) the second summand is equal to

$$\begin{aligned}
\lim_{s \rightarrow 0} \frac{1}{s} \int_0^1 \tau(VT_{f^{[1]}}^{D_r+srX, D_r}(srX)) dr &= \lim_{s \rightarrow 0} \int_0^1 \tau(VT_{f^{[1]}}^{D_r+srX, D_r}(rX)) dr \\
&= \int_0^1 \tau(VT_{f^{[1]}}^{D_r, D_r}(rX)) dr = \int_0^1 \tau(XT_{f^{[1]}}^{D_r, D_r}(V)) r dr,
\end{aligned}$$

where the second equality follows from Lemma 1.22 and the last equality follows from Lemma 3.2. Hence, by Lemma 1.11

$$\begin{aligned}
(A) &= \int_0^1 \tau(X[f(D_r) + rT_{f^{[1]}}^{D_r, D_r}(V)]) dr \\
&= \tau\left(X \int_0^1 [f(D_r) + rT_{f^{[1]}}^{D_r, D_r}(V)] dr\right),
\end{aligned}$$

where the integral on the RHS is a  $so^*$ -integral. By Proposition 1.20 and Lemma 1.22 the function  $r \in [0, 1] \mapsto f(D_r) + rT_{f^{[1]}}^{D_r, D_r}(V) \in \mathcal{L}^1(\mathcal{N}, \tau)$  is  $\mathcal{L}^1(\mathcal{N}, \tau)$ -continuous, so that the last integral

$$(B) := \int_0^1 [f(D_r) + rT_{f^{[1]}}^{D_r, D_r}(V)] dr$$

can be considered as Riemann integral. Let  $0 = r_0 < r_1 < \dots < r_n = 1$  be the partition of  $[0, 1]$  into  $n$  segments of equal length  $\frac{1}{n}$ . By the argument used in the proof of [3, Theorem 5.8] it can be shown that the  $\mathcal{L}^1(\mathcal{N}, \tau)$ -norm of  $T_{f^{[1]}}^{D_{r_j}, D_{r_j}}(V) - T_{f^{[1]}}^{D_r, D_r}(V)$ ,  $r \in [r_{j-1}, r_j]$ , has order  $\frac{1}{n}$ . Hence

$$\begin{aligned}
&\mathcal{L}^1\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \left( \frac{j}{n} T_{f^{[1]}}^{D_{r_j}, D_{r_j}}(V) - j \int_{r_{j-1}}^{r_j} T_{f^{[1]}}^{D_r, D_r}(V) dr \right) \\
&= \mathcal{L}^1\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n j \int_{r_{j-1}}^{r_j} (T_{f^{[1]}}^{D_{r_j}, D_{r_j}}(V) - T_{f^{[1]}}^{D_r, D_r}(V)) dr = 0,
\end{aligned}$$



so that by formula (12) applied to the pair  $(D_{r_{j-1}}, D_{r_j})$  we have

$$\begin{aligned}
(B) &= \mathcal{L}^1\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \left( f(D_{r_{j-1}}) + \frac{j}{n} T_{f^{[1]}}^{D_{r_j}, D_{r_j}}(V) \right) \\
&= \mathcal{L}^1\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n (f(D_{r_{j-1}}) + j(f(D_{r_j}) - f(D_{r_{j-1}}))) \\
&\quad + \mathcal{L}^1\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \left( \frac{j}{n} T_{f^{[1]}}^{D_{r_j}, D_{r_j}}(V) - j \int_{r_{j-1}}^{r_j} T_{f^{[1]}}^{D_r, D_r}(V) dr \right) \\
&= \mathcal{L}^1\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n (jf(D_{r_j}) - (j-1)f(D_{r_{j-1}})) = f(D_1).
\end{aligned}$$

□

The argument before [15, Proposition 1.5] now implies the following corollary.

**Corollary 3.6.** *The integral of the 1-form  $\alpha^f$  along a piecewise continuously differentiable path  $\Gamma$  in  $D_0 + \mathcal{N}$  depends only on the endpoints of the path  $\Gamma$ .*

**Proposition 3.7.** *If a self-adjoint operator  $D_0$  affiliated with  $\mathcal{N}$  has  $\tau$ -compact resolvent,  $D_1, D_2 \in D_0 + \mathcal{N}_{sa}$ , then for all  $\lambda \in \mathbb{R}$*

$$\xi_{D_2, D_0}(\lambda) = \xi_{D_2, D_1}(\lambda) + \xi_{D_1, D_0}(\lambda).$$

**Remark.** We emphasize that this additivity property is not almost everywhere in the spectral variable but in fact holds everywhere.

*Proof.* It follows from (19) that

$$\xi_{D_2, D_0}(\lambda) = \xi_{D_2, D_1}(\lambda) + \xi_{D_1, D_0}(\lambda) + C,$$

where  $C$  is a constant. Multiplying both sides of this equality by a positive  $C_c^2$ -function  $f$ , and integrating it, by Lemma 2.8 we get

$$\int_{\Gamma_{D_2, D_0}} \alpha^f = \int_{\Gamma_{D_2, D_1}} \alpha^f + \int_{\Gamma_{D_1, D_0}} \alpha^f + C \int_{\mathbb{R}} f(\lambda) d\lambda,$$

where  $\Gamma_{D_i, D_j}$  is the straight line path connecting operators  $D_i$  and  $D_j$ . The last equality and Corollary 3.6 imply that  $C = 0$ . □

**3.3. Spectral flow one-forms in the bounded case.** Since we obtain our unbounded spectral flow formula from a bounded one we need to study the map  $D \mapsto F_D = D(1 + D^2)^{-1/2}$  which takes the space of unbounded self adjoint operators with  $\tau$ -compact resolvent to the space  $\mathcal{F}^{-1,1}(\mathcal{N}, \tau)$  of bounded  $\tau$ -Fredholm operators  $F$  satisfying  $1 - F^2 \in \mathcal{K}(\mathcal{N}, \tau)$ .

Let  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ , let  $h \in C_c^2(a, b)$  and let  $K = F - F_0$ ,  $F_r := F_0 + rK$ . We define a 0-form  $\theta$  and a 1-form  $\alpha^h$  on the affine space  $\mathcal{A}_{F_0}$  by the formulae

$$\theta_F^h = \int_0^1 \tau(K h(F_r)) dr,$$

and

$$\alpha_F^h(X) = \tau(X h(F)), \quad X \in \mathcal{K}(\mathcal{N}, \tau).$$

By Lemmas 1.24 and 1.25, the operators  $h(F_r)$  and  $h(F)$  are  $\tau$ -trace class, so that the forms  $\theta^h$  and  $\alpha^h$  are well-defined.

**Proposition 3.8.** *If  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$  and if  $h \in C_c^2(a, b)$ , then*

$$d\theta_F^h(X) = \alpha_F^h(X),$$

where  $X \in \mathcal{K}(\mathcal{N}, \tau)$ , so that the 1-form  $\alpha_F^h$  is exact.

*Proof.* The proof follows verbatim the proof of Proposition 3.5, with references to Proposition 1.27 and Lemma 1.28 instead of Proposition 1.20 and Lemma 1.22.  $\square$

As in the unbounded case we get the following

**Corollary 3.9.** *The integral of the one-form  $\alpha^h$  depends only on the endpoints.*

**Corollary 3.10.** *Let  $F_j \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ ,  $j = 0, 1, 2$ , such that  $F_2 - F_1, F_1 - F_0 \in \mathcal{K}(\mathcal{N}, \tau)$ . Then for any  $\lambda \in (a, b)$  the following equality holds true*

$$\xi_{F_2, F_0}(\lambda) = \xi_{F_2, F_1}(\lambda) + \xi_{F_1, F_0}(\lambda).$$

We omit the proof as it is similar to that of Proposition 3.7.

Let

$$\varphi(\lambda) = \lambda (1 + \lambda^2)^{-1/2}.$$

It is easy to see that if  $D = D^* \eta \mathcal{N}$  is an operator with  $\tau$ -compact resolvent, then the operator  $F_D := \varphi(D)$  belongs to  $\mathcal{F}^{-1,1}(\mathcal{N}, \tau)$ .

**Proposition 3.11.** *If  $D_0 = D_0^* \eta \mathcal{N}$  is an operator with  $\tau$ -compact resolvent, and if  $V = V^* \in \mathcal{N}$ ,  $D_1 = D_0 + V$ , then the following equality holds*

$$\xi_{D_1, D_0}(\lambda) = \xi_{F_{D_1}, F_{D_0}}(\varphi(\lambda)).$$

*Proof.* Let  $h \in C_c^3(-1, 1)$  and  $f(\lambda) = h(\varphi(\lambda))$ . Then by Theorem 2.5

$$A := \tau(f(D_1) - f(D_0)) = \int_{\mathbb{R}} f'(\lambda) \xi_{D_1, D_0}(\lambda) d\lambda$$

and since  $F_{D_1} - F_{D_0} \in \mathcal{K}(\mathcal{N}, \tau)$  by [15, Lemma 2.7], we can apply Proposition 2.12 to get

$$\begin{aligned} A &= \tau(h(F_{D_1}) - h(F_{D_0})) = \int_{-1}^1 h'(t) \xi_{F_{D_1}, F_{D_0}}(t) dt \\ &= \int_{-\infty}^{\infty} h'(\varphi(\lambda)) \varphi'(\lambda) \xi_{F_{D_1}, F_{D_0}}(\varphi(\lambda)) d\lambda = \int_{-\infty}^{\infty} f'(\lambda) \xi_{F_{D_1}, F_{D_0}}(\varphi(\lambda)) d\lambda. \end{aligned}$$

Since  $f$  is an arbitrary  $C^2$ -function with compact support, comparing the last two formulas we get the equality

$$(31) \quad \xi_{D_1, D_0}(\lambda) = \xi_{F_{D_1}, F_{D_0}}(\varphi(\lambda)) + C.$$

It is left to show that the constant  $C = 0$ .

Let  $h$  be a non-negative function from  $C_c^\infty(-1, 1)$ . By Lemma 2.8 we have

$$(32) \quad \int_{\mathbb{R}} h(\varphi(\lambda)) \xi_{D_1, D_0}(\lambda) d\lambda = \int_0^1 \tau(Vh(F_{D_r})) dr.$$

Multiplying the first term of the RHS of (31) by  $h(\varphi(\lambda))$ , integrating it and using Lemma 2.15, we get

$$\begin{aligned} A &:= \int_{\mathbb{R}} h(\varphi(\lambda)) \xi_{F_{D_1}, F_{D_0}}(\varphi(\lambda)) d\lambda = \int_{-1}^1 h(\mu) \xi_{F_{D_1}, F_{D_0}}(\mu) (\varphi^{-1})'(\mu) d\mu \\ &= \int_0^1 \tau(Kh(F_r) (\varphi^{-1})'(F_r)) dr, \end{aligned}$$

where  $K = F_{D_1} - F_{D_0}$  and  $F_r$  is the straight line path connecting  $F_{D_1}$  and  $F_{D_0}$ . Let  $g(\mu) = h(\mu) (\varphi^{-1})'(\mu)$ . By Corollary 3.9 we have

$$A = \int_0^1 \tau(Kg(F_r)) dr = \int_0^1 \tau(\dot{F}_{D_r} g(F_{D_r})) dr.$$

By Proposition 1.18(ii) we have  $\dot{F}_{D_r} = T_{\varphi^{[1]}}^{D_r, D_r}(V)$ . Hence,

$$A = \int_0^1 \tau(T_{\varphi^{[1]}}^{D_r, D_r}(V) g(F_{D_r})) dr.$$

Using the BS-representation for  $\varphi^{[1]}$  given by (5), it follows from the definition of DOI (8), Lemma 1.12(ii) and Lemma 1.11, that

$$\begin{aligned}
(33) \quad A &= \int_0^1 \tau \left( \int_{\Pi} e^{i(s-t)D_r} V e^{itD_r} d\nu_{\varphi}(s, t) \cdot g(F_{D_r}) \right) dr \\
&= \int_0^1 \int_{\Pi} \tau \left( e^{i(s-t)D_r} V e^{itD_r} g(F_{D_r}) \right) d\nu_{\varphi}(s, t) dr \\
&= \frac{1}{\sqrt{2\pi}} \int_0^1 \int_{\mathbb{R}} \tau (V e^{isD_r} i s \hat{\varphi}(s) g(F_{D_r})) ds dr \\
&= \frac{1}{\sqrt{2\pi}} \int_0^1 \tau \left( V g(F_{D_r}) \int_{\mathbb{R}} e^{isD_r} i s \hat{\varphi}(s) ds \right) dr \\
&= \int_0^1 \tau (V g(F_{D_r}) \varphi'(D_r)) dr = \int_0^1 \tau (V h(F_{D_r})) dr,
\end{aligned}$$

since  $g(\varphi(\lambda))\varphi'(\lambda) = h(\varphi(\lambda))$ . It follows from (31), (32) and (33) that  $C \int_{\mathbb{R}} h(\varphi(\lambda)) d\lambda = 0$  and, hence,  $C = 0$ .  $\square$

**3.4. The first formula for spectral flow.** We establish first a spectral flow formula for bounded  $\tau$ -Fredholm operators. In this way we avoid a number of difficulties with unbounded operators. Then we make a ‘change of variable’ to get to the unbounded case.

First we require some additional notation which is important for establishing a convention for how we handle the situation when the endpoints have a kernel. Let  $a < 0$ ,  $b > 0$  and let  $\text{sign}_{a,b}$  be the function defined as  $\text{sign}_{a,b}(x) = b$  if  $x \geq 0$ , and  $\text{sign}_{a,b}(x) = a$  if  $x < 0$ .

We will write  $\tilde{F} = \text{sign}_{a,b}(F)$ , when it is clear from the context what the numbers  $a$  and  $b$  are.

**Definition 3.12.** If  $F \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$  and  $\kappa$  is a  $C^2$ -function on  $[0, \infty)$  vanishing in a neighbourhood of 0 then for  $h(\lambda) = \kappa((b - \lambda)(\lambda - a))$  we define  $\gamma_h(F)$  as

$$\gamma_h(F) = \int_0^1 \alpha_{F_r}^h(\dot{F}_r) dr,$$

where  $\alpha^h$  is the closed one-form defined before Proposition 3.8, and  $\{F_r\}_{r \in [0,1]}$  is the straight line connecting  $F$  and  $\tilde{F}$ .

The following theorem is the analogue in our setting of [16, Theorem 5.7]. It is the fundamental formula that we need as our starting point.

**Theorem 3.13.** Let  $F_0 \in \mathcal{F}^{a,b}(\mathcal{N}, \tau)$ , let  $K \in \mathcal{K}(\mathcal{N}, \tau)$  and let  $F_1 = F_0 + K$ . Let  $\kappa$  be a  $C^2$ -function on  $[0, \infty)$  vanishing in a neighbourhood of 0, such

that the integral of  $h(\lambda) = \kappa((b - \lambda)(\lambda - a))$  over  $(a, b)$  is equal to 1. Then the spectral flow between  $F_0$  and  $F_1$  is equal to

$$\text{sf}(F_0, F_1) = \int_a^b h(\lambda) \xi_{F_1, F_0}(\lambda) d\lambda + \gamma_h(F_1) - \gamma_h(F_0).$$

*Proof.* The proof follows ideas of [16, Theorem 5.7]. First of all by Phillips' definition of spectral flow [5] we have

$$\text{sf}(F_0, F_1) = \text{sf}(\tilde{F}_0, \tilde{F}_1).$$

(Note that, *a priori* we would expect to write

$$\text{sf}(F_0, F_1) = \text{sf}(F_0, \tilde{F}_0) + \text{sf}(\tilde{F}_0, \tilde{F}_1) + \text{sf}(\tilde{F}_1, F_1),$$

however there is no spectral flow along the paths joining  $F$  and  $\tilde{F}$  as noted in the proof of [15, Theorem 1.7].)

Now, by Lemma 3.1 we have

$$\text{sf}(\tilde{F}_0, \tilde{F}_1) = \int_0^1 \alpha_{\tilde{F}_r}^h(\dot{\tilde{F}}_r) dr,$$

where  $\{\tilde{F}_r\}_{r \in [0,1]}$  is the straight line path, connecting  $\tilde{F}_0$  and  $\tilde{F}_1$ . By Corollary 3.9 we can replace this path by the (broken) path given on this diagram

$$\begin{array}{ccc} F_0 & \dashrightarrow & F_1 \\ \uparrow & & \downarrow \\ -\gamma_h(F_0) & & \gamma_h(F_1) \\ \downarrow & & \downarrow \\ \tilde{F}_0 & \longrightarrow & \tilde{F}_1 \end{array}$$

Then we get

$$\text{sf}(\tilde{F}_0, \tilde{F}_1) = -\gamma_h(F_0) + \int_0^1 \alpha_{\tilde{F}_r}^h(\dot{\tilde{F}}_r) dr + \gamma_h(F_1),$$

where  $\{F_r\}_{r \in [0,1]}$  is the straight line path, connecting  $F_0$  and  $F_1$ . But, setting  $F_1 - F_0 = K$ , we have by Lemma 2.15

$$\int_0^1 \alpha_{\tilde{F}_r}^h(\dot{\tilde{F}}_r) dr = \int_0^1 \tau(Kh(F_r)) dr = \int_{\mathbb{R}} h(\lambda) \xi_{F_1, F_0}(\lambda) d\lambda.$$

□

**Theorem 3.14.** *Let  $F_0 \in \mathcal{F}^{-1,1}(\mathcal{N}, \tau)$ , let  $K \in \mathcal{K}(\mathcal{N}, \tau)$  and let  $F_1 = F_0 + K$ . Let  $\kappa$  be a  $C^2$ -function on  $[0, \infty)$  vanishing in a neighbourhood of 0, such that the integral of  $h(\lambda) = \kappa(1 - \lambda^2)$  over  $(-1, 1)$  is equal to 1. Then the spectral flow function for the pair  $F_0$  and  $F_1$  is equal to*

$$\text{sf}(\mu; F_0, F_1) = \int_{-1}^1 h(\lambda) \xi_{F_1, F_0}(\lambda) d\lambda + \gamma_{h-\mu}(F_1 - \mu) - \gamma_{h-\mu}(F_0 - \mu),$$

where  $h_{-\mu}(\lambda) = h(\lambda + \mu)$ .

*Proof.* By definition we have

$$\text{sf}(\mu; F_0, F_1) = \text{sf}(F_0 - \mu, F_1 - \mu).$$

Since  $F_j - \mu \in \mathcal{F}^{-1-\mu, 1-\mu}(\mathcal{N}, \tau)$ , by previous theorem we have

$$\begin{aligned} \text{sf}(F_0 - \mu, F_1 - \mu) &= \int_{-1-\mu}^{1-\mu} h_{-\mu}(\lambda) \xi_{F_1-\mu, F_0-\mu}(\lambda) d\lambda + \gamma_{h_{-\mu}}(F_1 - \mu) - \gamma_{h_{-\mu}}(F_0 - \mu). \end{aligned}$$

Since  $\xi_{F_1-\mu, F_0-\mu}(\lambda) = \xi_{F_1, F_0}(\lambda + \mu)$ , we have

$$\begin{aligned} \text{sf}(F_0 - \mu, F_1 - \mu) &= \int_{-1-\mu}^{1-\mu} h(\lambda + \mu) \xi_{F_1, F_0}(\lambda + \mu) d\lambda + \gamma_{h_{-\mu}}(F_1 - \mu) - \gamma_{h_{-\mu}}(F_0 - \mu) \\ &= \int_{-1}^1 h(\lambda) \xi_{F_1, F_0}(\lambda) d\lambda + \gamma_{h_{-\mu}}(F_1 - \mu) - \gamma_{h_{-\mu}}(F_0 - \mu). \end{aligned}$$

□

**Corollary 3.15.** *If  $F_0$  and  $F_1$  are unitarily equivalent, then*

$$\text{sf}(\mu; F_0, F_1) = \xi_{F_1, F_0}(\mu) = \text{const}.$$

*Proof.* By Corollary 2.13 the function  $\xi_{F_1, F_0}(\cdot)$  is constant on  $(-1, 1)$ , so that  $\int_{\mathbb{R}} h(\lambda) \xi_{F_1, F_0}(\lambda) d\lambda = \xi_{F_1, F_0}(0)$ .

If  $F_0$  and  $F_1$  are unitarily equivalent, then  $\gamma_{h_{-\mu}}(F_1 - \mu) = \gamma_{h_{-\mu}}(F_0 - \mu)$ . Hence, for all  $\mu \in (-1, 1)$

$$\text{sf}(\mu; F_0, F_1) = \xi_{F_1, F_0}(\mu) = \xi_{F_1, F_0}(0).$$

□

**Lemma 3.16.** *If  $F \in \mathcal{F}^{-1, 1}(\mathcal{N}, \tau)$  and if  $\{h_\varepsilon\}_{\varepsilon > 0}$  is an approximate  $\delta$  function (by compactly supported even functions) then for all  $\mu \in (-1, 1)$  the limit*

$$\gamma_\mu(F) := \lim_{\varepsilon \rightarrow 0} \gamma_{h_\varepsilon}(F - \mu)$$

*exists and is equal to  $\xi_{G, \tilde{G}}(0)$ , where  $G = F - \mu$ .*

*Proof.* Since  $h$  is an even function we have that

$$\int_{-\infty}^0 h_\varepsilon(\lambda) \xi_{G, \tilde{G}}(\lambda) d\lambda \rightarrow \frac{1}{2} \xi_{G, \tilde{G}}(0-)$$

and

$$\int_0^{\infty} h_\varepsilon(\lambda) \xi_{G, \tilde{G}}(\lambda) d\lambda \rightarrow \frac{1}{2} \xi_{G, \tilde{G}}(0+),$$

as  $\varepsilon \rightarrow 0$ . If  $\{G_r\}_{r \in [0,1]}$  is the straight line path connecting  $G$  and  $\tilde{G}$  then by Lemma 2.15 we have

$$\begin{aligned} \gamma_{h_\varepsilon}(G) &= \int_0^1 \alpha_{h_\varepsilon}(\dot{G}_r) dr = \int_0^1 \tau \left( \dot{G}_r h_\varepsilon(G_r) \right) dr \\ &= \int_{\mathbb{R}} h_\varepsilon(\lambda) \xi_{G, \tilde{G}}(\lambda) d\lambda \rightarrow \frac{1}{2} \left( \xi_{G, \tilde{G}}(0-) + \xi_{G, \tilde{G}}(0+) \right) = \xi_{G, \tilde{G}}(0) \end{aligned}$$

as  $\varepsilon \rightarrow 0$ , by Definition 2.6 of  $\xi$  at discontinuity points.  $\square$

Now we need to handle the situation when the endpoints are not unitarily equivalent. For this we require some additional facts about the ‘end-point correction terms’. The interesting fact which we now establish is that the approach using the spectral shift function differs in a fundamental way from the previous point of view in [16]. The next few results demonstrate this by showing that the spectral shift function absorbs the contribution to the formula due to the spectral asymmetry of the endpoints leaving only kernel correction terms to be handled.

**Lemma 3.17.** *If  $F \in \mathcal{F}^{-1,1}(\mathcal{N}, \tau)$  and if  $\mu \in (-1, 1)$ , then the following equality holds true*

$$\gamma_\mu(F) = \frac{1}{2} \tau[\ker(F - \mu)].$$

*Proof.* Let  $G = F - \mu$ . We have

$$\tau[\ker G] = \tau \left( E_{(-\infty, 0]}^G - E_{(-\infty, 0]}^{\tilde{G}} \right).$$

By Proposition 2.12(ii) and Definition 2.14 the value  $\xi_{\tilde{G}, G}(0)$  is the half sum of the last expression and

$$\tau \left( E_{(-\infty, 0)}^G - E_{(-\infty, 0)}^{\tilde{G}} \right) = 0.$$

Hence, by Lemma 3.16

$$\gamma_\mu(F) = \xi_{G, \tilde{G}}(0) = \frac{1}{2} \tau([\ker G]).$$

$\square$

**Theorem 3.18.** *If  $F_0, F_1 \in \mathcal{F}^{-1,1}(\mathcal{N}, \tau)$  such that  $F_1 - F_0 \in \mathcal{K}(\mathcal{N}, \tau)$ , then for all  $\mu \in (-1, 1)$*

$$(34) \quad \text{sf}(\mu; F_0, F_1) = \xi_{F_1, F_0}(\mu) + \frac{1}{2} (\tau[\ker(F_1 - \mu)] - \tau[\ker(F_0 - \mu)]).$$

*Proof.* Replace  $h$  in Theorem 3.14 by  $h_{\varepsilon, \mu}$  (thus translate the approximate  $\delta$  function  $h_\varepsilon$  by  $\mu$ ) and then let  $\varepsilon \rightarrow 0$  using Lemmas 3.16, 3.17.  $\square$

We now see that under hypotheses that guarantee both are defined the spectral flow function and the spectral shift function differ only by kernel corrections terms for the endpoints. We should remark that the occurrence of the correction terms  $\gamma_\mu(F_j)$ ,  $j = 1, 2$ , in the last formula can be explained by the fact that we actually define the spectral flow function and the spectral shift function at discontinuity points in different ways. The spectral shift function is defined as a half-sum of the left and the right limits, while the spectral flow is defined to be left-continuous.

**3.5. Spectral flow in the unbounded case.** The formulae for spectral flow in the bounded case may be now used to establish corresponding results in our original setting of unbounded self adjoint operators with compact resolvent.

By Proposition 3.11  $\xi_{D_1, D_0}(0) = \xi_{F_{D_1}, F_{D_0}}(0)$  and by definition of spectral flow for unbounded operators [5]  $\text{sf}(D_0, D_1) = \text{sf}(F_{D_1}, F_{D_0})$ . Hence, it follows from (34) taken at  $\mu = 0$  that

$$\text{sf}(D_0, D_1) = \xi_{D_1, D_0}(0) + \gamma_0(F_1) - \gamma_0(F_0).$$

Since

$$\ker(D) = \ker(F_D)$$

we have the following equality

$$\text{sf}(D_0, D_1) = \xi_{D_1, D_0}(0) + \frac{1}{2}\tau[\ker(D_1)] - \frac{1}{2}\tau[\ker(D_0)].$$

If we replace here the operators  $D_0$  and  $D_1$  by the operators  $D_0 - \lambda$  and  $D_1 - \lambda$  respectively then we get

$$\text{sf}(\lambda; D_0, D_1) = \xi_{D_1 - \lambda, D_0 - \lambda}(0) + \frac{1}{2}\tau[\ker(D_1 - \lambda)] - \frac{1}{2}\tau[\ker(D_0 - \lambda)].$$

Since  $\xi_{D_1 - \lambda, D_0 - \lambda}(0) = \xi_{D_1, D_0}(\lambda)$  it follows that

$$(35) \quad \text{sf}(\lambda; D_0, D_1) = \xi_{D_1, D_0}(\lambda) + \frac{1}{2}\tau[\ker(D_1 - \lambda)] - \frac{1}{2}\tau[\ker(D_0 - \lambda)].$$

**3.6. The spectral flow formula using infinitesimal spectral flow.** The results on the spectral shift function that we established in Section 2 now suggest a new direction for spectral flow theory.

**Definition 3.19.** Let  $D_0$  be a self-adjoint operator affiliated with  $\mathcal{N}$  having  $\tau$ -compact resolvent. The infinitesimal spectral flow one-form is a distribution-valued one-form  $\Phi_D$  on the affine space  $D_0 + \mathcal{N}_{sa}$ , defined by formula

$$\langle \Phi_D(X), \varphi \rangle = \tau(X\varphi(D)), \quad X \in \mathcal{N}_{sa}, \quad \varphi \in C_c^\infty(\mathbb{R}).$$

Formally,

$$\Phi_D(X) = \tau(X\delta(D)),$$

where  $\delta(D)$  is the  $\delta$ -function of  $D$ .



We believe that the notion of infinitesimal spectral flow will have application to the problem of studying spectral flow when the endpoints differ by a  $D_0$  relatively bounded perturbation. To this end we establish that spectral flow may be reformulated in terms of it.

**Theorem 3.20.** *Let  $D_1 \in D_0 + \mathcal{N}_{sa}$ . Spectral flow between  $D_0$  and  $D_1$  is equal to the integral of the infinitesimal spectral flow one-form along any piecewise  $C^1$ -path  $\{D_r\}_{r \in [0,1]}$  in  $D_0 + \mathcal{N}$  connecting  $D_0$  and  $D_1$  in the sense that for any  $\varphi \in C_c^\infty(\mathbb{R})$  the following equality holds true*

$$\int_{\mathbb{R}} \text{sf}(\lambda; D_0, D_1) \varphi(\lambda) d\lambda = \int_0^1 \langle \Phi_{D_r}(\dot{D}_r), \varphi \rangle dr.$$

Formally,

$$\text{sf}(D_0, D_1) = \int_0^1 \Phi_{D_r}(\dot{D}_r) dr, \quad \text{or} \quad \text{sf}(\lambda; D_0, D_1) = \int_0^1 \Phi_{D_r - \lambda}(\dot{D}_r) dr.$$

*Proof.* By Corollary 3.6 we can choose the path  $\{D_r\}_{r \in [0,1]}$  to be the straight line path  $D_r = D_0 + rV$ . It follows from Lemmas 1.3 and 1.4 that the functions  $\lambda \mapsto \tau([\ker(D_0 - \lambda)])$  and  $\lambda \mapsto \tau([\ker(D_1 - \lambda)])$  can be non-zero only on a countable set. Hence, by (35) and Lemma 2.8 we have

$$\begin{aligned} \int_{\mathbb{R}} \text{sf}(\lambda; D_0, D_1) \varphi(\lambda) d\lambda &= \int_{\mathbb{R}} \xi_{D_1, D_0}(\lambda) \varphi(\lambda) d\lambda \\ &= \int_0^1 \tau(V\varphi(D_r)) dr = \int_0^1 \langle \Phi_{D_r}(\dot{D}_r), \varphi \rangle dr. \end{aligned}$$

□

We remark that the infinitesimal spectral flow one-form is exact in the sense that its value on every test function is exact.

**3.7. The spectral flow formulae in the  $\mathcal{I}$ -summable spectral triple case.** The original approach of [16] required summability constraints on the operator  $D_0$ . We will now see that if indeed  $D_0$  satisfies such conditions then we can weaken conditions on the function  $f$  in Theorem 2.10.

**Lemma 3.21.** *Let  $D_0$  be a self-adjoint operator with  $\tau$ -compact resolvent affiliated with  $\mathcal{N}$ . Let  $g$  be an increasing continuous function on  $[0, +\infty)$ , such that  $g(0) \geq 0$  and  $g(c(1 + D^2)^{-1}) \in \mathcal{L}^1(\mathcal{N}, \tau)$  for all  $c > 0$ . Let  $f(x) = g((1 + x^2)^{-1})$ . Then for any  $R > 0$  and for any  $V = V^* \in \mathcal{N}$  the operator  $f(D + V)$  is trace class and the function*

$$V \in B_R \mapsto \|f(D + V)\|_1$$

*is bounded.*

*Proof.* By [22, Lemma 2.5(iv)] we have for all  $t > 0$

$$\mu_t(f(D + V)) = \mu_t\left(g\left((1 + (D + V)^2)^{-1}\right)\right) = g\left(\mu_t\left((1 + (D + V)^2)^{-1}\right)\right).$$

By Lemma 1.7 there exists a constant  $c = c(R) > 0$  such that for any  $V \in B_R$

$$(1 + (D + V)^2)^{-1} \leq c(1 + D^2)^{-1}.$$

Hence, by [22, Lemma 2.5(iii)] we have

$$\mu_t(f(D + V)) \leq g\left(\mu_t\left[c(1 + D^2)^{-1}\right]\right) = \mu_t\left(g\left[c(1 + D^2)^{-1}\right]\right).$$

Since  $g(c(1 + D^2)^{-1}) \in \mathcal{L}^1(\mathcal{N}, \tau)$ , the last function belongs to  $L^1[0, \infty)$ , which implies that  $f(D + V) \in \mathcal{L}^1(\mathcal{N}, \tau)$ .  $\square$

**Lemma 3.22.** *Let  $D_0$ ,  $g$  and  $f$  be as in Lemma 3.21. An integral of the one-form*

$$\alpha_D^f(X) = \tau(Xf(D)), \quad X \in \mathcal{N}, \quad D \in D_0 + \mathcal{N}_{sa},$$

*along a piecewise smooth path in  $D_0 + \mathcal{N}_{sa}$  depends only on endpoints of that path.*

*Proof.* Let  $f_n$  be a increasing sequence of compactly supported smooth functions converging pointwise to  $f$  and  $\Gamma_1, \Gamma_2$  be two piecewise smooth paths in  $D_0 + \mathcal{N}_{sa}$  with the same endpoints. Then by Lemma 3.21, Lebesgue dominated convergence theorem and Corollary 3.6 we have

$$\begin{aligned} \int_{\Gamma_1} \alpha^f &= \int_{\Gamma_1} \lim_{n \rightarrow \infty} \alpha^{f_n} = \lim_{n \rightarrow \infty} \int_{\Gamma_1} \alpha^{f_n} \\ &= \lim_{n \rightarrow \infty} \int_{\Gamma_2} \alpha^{f_n} = \int_{\Gamma_2} \lim_{n \rightarrow \infty} \alpha^{f_n} = \int_{\Gamma_2} \alpha^f. \end{aligned}$$

$\square$

The condition that  $g(c(1 + D_0^2)^{-1})$  be trace class is a generalised summability constraint. This notion arises naturally for certain ideals  $\mathcal{I}$  of compact operators (for example for the Schatten ideals  $\mathcal{I}_p$ ,  $p \geq 1$ ,  $g(x) = x^{p/2}$  and we have the notion of  $p$ -summability). We have also already remarked on the  $\theta$ -summable case.

Now if there is a unitary  $u \in \mathcal{N}$  with  $V = u^*[D_0, u]$  bounded then we have, for a dense subalgebra  $\mathcal{A}$  of the  $C^*$ -algebra generated by  $u$ , a semifinite ‘ $g$ -summable’ spectral triple  $(\mathcal{A}, \mathcal{N}, D_0)$ . Moreover  $D_0 + V = uD_0u^*$  so we have unitarily equivalent endpoints.

**Theorem 3.23.** *Let  $f$  be a non-negative  $L^1$ -function such that  $f(D_r) \in \mathcal{L}^1(\mathcal{N}, \tau)$  for all  $r \in [0, 1]$ , and let  $r \mapsto \|f(D_r)\|_1$  be integrable on  $[0, 1]$ . If*

$D_0$  and  $D_1$  are unitarily equivalent then

$$\text{sf}(\lambda; D_0, D_1) = C^{-1} \int_0^1 \tau(Vf(D_r - \lambda)) \, dr,$$

where  $C = \int_{-\infty}^{\infty} f(\lambda) \, d\lambda$ .

*Proof.* Unitary equivalence of  $D_0$  and  $D_1$  implies that two last terms in (35) vanish. In case of  $f \in B_c(\mathbb{R})$ , multiplying (35) by  $f(\lambda)$  and integrating it we get the required equality by Lemma 2.8 and Theorem 2.9. For an arbitrary  $f \in L^1$  the claim follows from Lebesgue's dominated convergence theorem by approximating  $f$  by an increasing sequence of step-functions converging a.e. to  $f$ .  $\square$

The following corollary recovers two of the main results of [15, 16].

**Corollary 3.24.** (i) If  $D_0$  is  $\theta$ -summable with respect to  $\mathcal{N}$  and if  $D_0$  and  $D_1$  are unitarily equivalent then

$$\text{sf}(D_0, D_1) = \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau(Ve^{-\varepsilon D_r^2}) \, dr.$$

(ii) If  $D_0$  is  $p$ -summable (i.e.  $(1 + D_0^2)^{-p/2} \in \mathcal{L}^1(\mathcal{N}, \tau)$ ) with respect to  $\mathcal{N}$ , where  $p > 1$  and if  $D_0$  and  $D_1$  are unitarily equivalent then

$$\text{sf}(D_0, D_1) = C_p^{-1} \int_0^1 \tau(V(1 + D_r^2)^{-\frac{p}{2}}) \, dr,$$

where  $C_p = \int_{-\infty}^{\infty} (1 + \lambda^2)^{-\frac{p}{2}} \, d\lambda$ .

*Proof.* Put  $f(\lambda) = e^{-\varepsilon \lambda^2}$  and  $f(\lambda) = (1 + \lambda^2)^{-\frac{p}{2}}$  for (i) and (ii) respectively in Theorem 3.23. The conditions of that theorem are fulfilled by Lemma 3.21.  $\square$

**3.8. Recovering  $\eta$ -invariants.** To demonstrate that we have indeed generalized previous analytic approaches to spectral flow formulae we still need some refinements. What is missing is the relationship of the ‘end-point correction terms’ to the truncated eta invariants of [24].

In fact Theorem 3.13 combined with some ideas of [16] will now enable us to give a new proof of the original formula (28) for spectral flow with unitarily inequivalent endpoints.

Introduce the function

$$\kappa_\varepsilon(\lambda) = \sqrt{\frac{\varepsilon}{\pi}} \lambda^{-3/2} e^{\varepsilon(1-\lambda^{-1})}.$$

and let  $h_\varepsilon(\lambda) = \kappa_\varepsilon(1 - \lambda^2)$ ,  $f_\varepsilon(\lambda) = \kappa_\varepsilon((1 + \lambda^2)^{-1})$ .

**Lemma 3.25.** *Let  $D_0 = D_0^* \eta \mathcal{N}$  be  $\theta$ -summable, let  $V \in \mathcal{N}_{sa}$  and let  $D_1 = D_0 + V$ . Then*

$$\int_{-1}^1 h_\varepsilon(\lambda) \xi_{F_{D_1}, F_{D_0}}(\lambda) d\lambda = \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau \left( V e^{-\varepsilon D_r^2} \right) dr.$$

*Proof.* Since  $h_\varepsilon(\varphi(\mu)) = f_\varepsilon(\mu)$ , by Proposition 3.11 we have

$$\begin{aligned} (A) &:= \int_{-1}^1 h_\varepsilon(\lambda) \xi_{F_{D_1}, F_{D_0}}(\lambda) d\lambda = \int_{-\infty}^{\infty} h_\varepsilon(\varphi(\mu)) \xi_{F_{D_1}, F_{D_0}}(\varphi(\mu)) \varphi(\mu)' d\mu \\ &= \int_{-\infty}^{\infty} f_\varepsilon(\mu) \varphi'(\mu) \xi_{D_1, D_0}(\mu) d\mu. \end{aligned}$$

Further, by Lemmas 2.4 and 2.3

$$\begin{aligned} (A) &= \int_{-\infty}^{\infty} f_\varepsilon(\mu) \varphi'(\mu) \int_0^1 \tau(V dE_\mu^{D_r}) dr = \int_0^1 \tau(V f_\varepsilon(D_r) \varphi'(D_r)) dr \\ &= \int_0^1 \tau \left( V \sqrt{\frac{\varepsilon}{\pi}} (1 + D_r^2)^{3/2} e^{-\varepsilon D_r^2} (1 + D_r^2)^{-3/2} \right) dr \\ &= \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau \left( V e^{-\varepsilon D_r^2} \right) dr. \end{aligned}$$

□

As we have emphasized previously, the strategy of our proof follows that of [16] in that, we deduce the unbounded version of the spectral flow formula for the theta summable case from a bounded version. To this end introduce  $F_s = D(s + D^2)^{-1/2}$ .

**Lemma 3.26.** [16, Lemma 8.8] *We have*

$$\lim_{\delta \rightarrow 0} \int_{\Gamma_\delta} \alpha^{h_\varepsilon} = 0,$$

where  $\Gamma_\delta$  is the straight line connecting  $F_0$  and  $F_\delta$ .

**Lemma 3.27.** *If  $D = D^* \eta \mathcal{N}$  is  $\theta$ -summable, then the following equality holds true*

$$\gamma_{h_\varepsilon}(F_D) = \frac{1}{2} (\eta_\varepsilon(D) + \tau[\ker D]).$$

*Proof.* We note that  $1 - F_s^2 = s(s + D^2)^{-1}$  and that  $\dot{F}_s = -\frac{1}{2} D(s + D^2)^{-3/2}$ . The path  $\Gamma_1 := \{F_s\}_{s \in [0,1]}$  connects  $\text{sgn}(F_D)$  with  $F_D$ . If we denote by  $\Gamma_2$  the straight line path connecting  $\text{sgn}(F_D)$  with  $\tilde{F} = \text{sign}(F_D)$  then the path  $-\Gamma_1 + \Gamma_2$  connects  $F_D$  with  $\tilde{F}$ , so that by Lemma 3.22 applied to  $f = h_\varepsilon$ ,

and by the argument of [16] and Lemma 3.26 dealing with discontinuity of the path  $\Gamma_1$  at zero, it follows that

$$\gamma_{h_\varepsilon}(F_D) = - \int_{\Gamma_1} \alpha^{h_\varepsilon} + \int_{\Gamma_2} \alpha^{h_\varepsilon}.$$

We have for the first summand

$$\begin{aligned} \int_{\Gamma_1} \alpha^{h_\varepsilon} &= \int_0^1 \alpha_{F_s}^{h_\varepsilon}(\dot{F}_s) ds = \int_0^1 \tau \left( \dot{F}_s h_\varepsilon(F_s) \right) ds \\ &= -\frac{1}{2} \int_0^1 \tau \left( D(s + D^2)^{-3/2} \kappa_\varepsilon(1 - F_s^2) \right) ds \\ &= -\frac{1}{2} \int_0^1 \tau \left( D(s + D^2)^{-3/2} \kappa_\varepsilon(s(s + D^2)^{-1}) \right) ds \\ &= -\frac{1}{2} \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau \left( D(s + D^2)^{-3/2} s^{-3/2} (s + D^2)^{3/2} e^{-\frac{\varepsilon}{s} D^2} \right) ds \\ &= -\frac{1}{2} \sqrt{\frac{\varepsilon}{\pi}} \int_0^1 \tau \left( D e^{-\frac{\varepsilon}{s} D^2} \right) s^{-3/2} ds \\ &= -\frac{1}{2} \sqrt{\frac{\varepsilon}{\pi}} \int_1^\infty \tau \left( D e^{-\varepsilon t D^2} \right) \frac{dt}{\sqrt{t}} = -\frac{1}{2} \eta_\varepsilon(D). \end{aligned}$$

Let  $E = [\ker D]$  and let  $G_r = \operatorname{sgn}(F_D) + rE$  be the path  $\Gamma_2$ . Then the second summand is equal to

$$\int_{\Gamma_2} \alpha^{h_\varepsilon} = \int_0^1 \tau \left( \dot{G}_r h_\varepsilon(G_r) \right) dr = \int_0^1 \tau \left( E \kappa_\varepsilon(1 - G_r^2) \right) dr.$$

Since  $1 - G_r^2 = E(1 - r^2)$ , it follows that

$$\int_{\Gamma_2} \alpha^{h_\varepsilon} = \int_0^1 \tau \left( E \kappa_\varepsilon(1 - r^2) \right) dr = \tau(E) \int_0^1 \kappa_\varepsilon(1 - r^2) dr = \frac{1}{2} \tau(E),$$

so that  $\gamma_{h_\varepsilon}(F_D) = \frac{1}{2} (\eta_\varepsilon(D) + \tau[\ker D])$ .  $\square$

As a direct corollary of these Lemmas and Theorem 3.13 we get a new proof of (28). The idea used in the proof of the next theorem, of approximating the exponential function by functions of compact support, has been exploited in the context of spectral flow formula in [35].

**Theorem 3.28.** *If  $D_0$  is  $\theta$ -summable then the formula (28) holds true.*

*Proof.* Let  $h_n$  be a sequence of smooth non-negative functions, compactly supported on  $(-1, 1)$ , and converging pointwise to  $h_\varepsilon$ . Recall that  $D_r = D_0 + rV$ . Then the sequence  $\gamma_{h_n}(F_{D_r})$  converges to  $\gamma_{h_\varepsilon}(F_{D_r})$  and the sequence  $\int_{-1}^1 h_n(\lambda) \xi_{F_{D_1}, F_{D_0}}(\lambda) d\lambda$  converges to  $\int_{-1}^1 h_\varepsilon(\lambda) \xi_{F_{D_1}, F_{D_0}}(\lambda) d\lambda$  by Lebesgue's DCT, since  $\theta$ -summability of  $D_0$  implies 1-summability of  $h_\varepsilon(F_{D_r})$ . Hence the claim follows from Theorem 3.13, Lemma 3.25 and Lemma 3.27.  $\square$

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